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Dynamic stability of pre-twisted beams with non-constant spin rates under axial random forces ☆

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Abstract

This paper investigates the dynamic stability of a pre-twisted cantilever beam spinning along its longitudinal axis with a periodically varying speed and acted upon by an axial random force at the free end. The spin rate of the beam is characterized as a small periodic perturbation superimposed on a constant speed, and the axial force is assumed as the sum of a static force and a weakly stationary random process with a zero mean. Both the periodically varying spin rate and the axial random force may lead to parametric instability of the beam. In this work, the finite element method is applied first to get rid of the dependence on the spatial coordinate. The method of stochastic averaging is then adopted to obtain Ito's equations for the system response under different resonant frequency combinations. Finally, the first-moment and the second-moment stability conditions of the beam are derived explicitly. Numerical results are presented for a simple harmonic speed perturbation and a Gaussian white noise axial force.

Keywords: Dynamic; Stability; Random forces

1. Introduction

Drilling is one of the most widely used operations in manufacturing. In a drilling process, cutting parameters, such as drill geometry, spin speed, thrust force, etc., will affect wear and breakage of the drill and accuracy of the hole produced. Therefore, the dynamic behavior of drills has been an interesting topic for research.

A standard twist drill looks like a solid bar having a length-to-diameter ratio about 10 or even higher. It usually has two helical flutes and can be regarded as a pre-twisted beam with unequal flexural rigidity in two orthogonal directions. Lots of references pertaining to the vibration of pre-twisted beams can be found in the literature. Most of the works (Carnegie and Thomas, 1972; Rao, 1972; Swaminathan and Rao, 1977;

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Subrahmanyam et al., 1981; Subrahmanyam and Rao, 1982) concerned with the free vibration of turbine blades and propellers, which are treated as pre-twisted, tapered cantilever beams rotating about the axis perpendicular to the longitudinal axis of the beam.

When dealing with the dynamic behavior of drills, the axial force is an important factor and has to be taken into consideration. Magrab and Gilsinn (1984) calculated the natural frequencies of a clamped–clamped pre-twisted beam under a static axial force by the Galerkin method. Tekinalp and Ulsoy (1989, 1990) investigated the free vibration of a spinning pre-twisted beam subjected to a static axial force by the finite element method. An extensive study of the elastic stability of spinning, pre-twisted beams acted upon by conservative axial forces was conducted by Liao and Dang (1992).

In fact, the axial force acting on drills usually fluctuates within a small range of variation during service. Liao and Huang (1995a) analyzed the parametric stability of spinning pre-twisted beams under periodic axial forces. Summed-type resonance is shown to exist due to this periodic axial force. Young and Gau (2003) extended Liao and Huang's efforts further to investigate the parametric random stability of spinning pre-twisted beams subjected to axial random forces. Numerical results are given for a Gaussian white noise excitation, and the effects of various system parameters on the mean-square stability boundary of the beam are illustrated.

The spin rates of the pre-twisted beams considered in the above-mentioned references are constant. In reality, the spin rate often varies within a small speed interval for most spinning objects under external disturbances. Therefore, it would be more general and physically realistic to consider a time-dependent spin rate for spinning objects. Kammer and Schlack (1987a,b) appeared to be the first ones to consider the dynamic behavior of a uniform beam with a periodically varying angular speed. Summed-type resonance is shown to exist due to this periodically varying angular speed. Dynamic responses of various rotating structures with non-constant speeds were investigated by Young and his co-worker (1991, 1992, 1993). Later Liao and Huang (1995b) also studied the parametric stability of pre-twisted beams spinning with periodically varying speeds and subjected to static axial forces.

Therefore, this work extends the past efforts of the authors further to analyze the parametric random stability of pre-twisted beams spinning with periodically varying speeds and acted upon by axial random forces. Due to the periodically varying spin rate and the axial random force, the pre-twisted beam is subjected to both parametric and parametric random excitations simultaneously. The method of stochastic averaging (Sri Namachchivaya, 1989) is used to deal with both deterministic and random excitations at the same time to obtain the system response.

2. Equations of motion

Consider a pre-twisted, cantilever beam of length L spinning along its longitudinal axis with a spin rate Ω and subjected to an axial force P at its free end, as shown in Fig. 1. In this figure, (X, Y, Z) is a fixed coordinate system, while (x, y, z) is a rotating coordinate system attached to the beam with the x-axis aligned with the X-axis. The (x', y', z') coordinate system rotates along the longitudinal axis of the beam with a total pre-twisted angle γ such that the y'- and z'-axes coincide with the principal axes of the beam at every cross-section.

If every cross-section of the beam is symmetric with respect to two principal axes of inertia, torsional coupling will not be presented, and only flexural bending is about to occur. In addition, flexural bending takes place simultaneously in two mutually perpendicular planes with unequal flexural rigidity in these two principal axes, and coupling arises due to the presence of the pre-twisted angle (Swaminathan and Rao, 1977). Thus the equations of motion and the boundary conditions for the pre-twisted, cantilever beam can be derived as follows (Liao and Huang, 1995b):

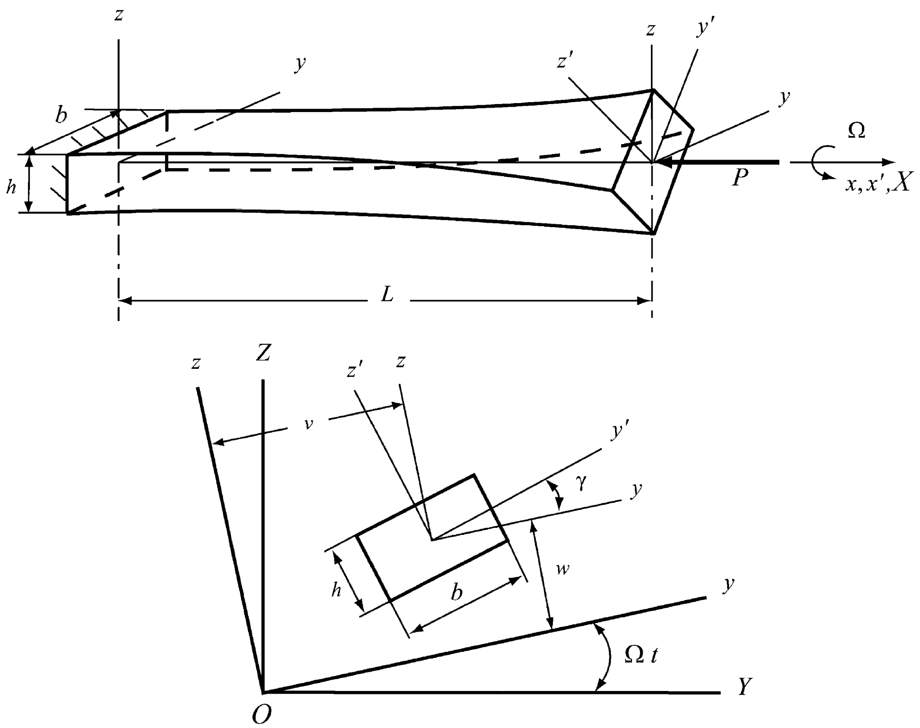

Fig. 1. A spinning pre-twisted cantilever beam subjected to an axial force at the free end.

Equations of motion

$$\begin{aligned}
&\rho\left[A\left(\frac{\partial^2 v}{\partial t^2}-2\Omega\frac{\partial w}{\partial t}-w\frac{\partial \Omega}{\partial t}-\Omega^2 v\right)-\frac{\partial}{\partial x}\left(I_{zz}\frac{\partial^3 v}{\partial t^2\partial x}+I_{yz}\frac{\partial^3 w}{\partial t^2\partial x}\right)\right]\\
&\quad+c\frac{\partial v}{\partial t}+E\frac{\partial^2}{\partial x^2}\left(I_{zz}\frac{\partial^2 v}{\partial x^2}+I_{yz}\frac{\partial^2 w}{\partial x^2}\right)+P(t)\frac{\partial^2 v}{\partial x^2}=0,\\
&\rho\left[A\left(\frac{\partial^2 w}{\partial t^2}+2\Omega\frac{\partial v}{\partial t}+v\frac{\partial \Omega}{\partial t}-\Omega^2 w\right)-\frac{\partial}{\partial x}\left(I_{yz}\frac{\partial^3 v}{\partial t^2\partial x}+I_{yy}\frac{\partial^3 w}{\partial t^2\partial x}\right)\right]\\
&\quad+c\frac{\partial w}{\partial t}+E\frac{\partial^2}{\partial x^2}\left(I_{yz}\frac{\partial^2 v}{\partial x^2}+I_{yy}\frac{\partial^2 w}{\partial x^2}\right)+P(t)\frac{\partial^2 w}{\partial x^2}=0,
\end{aligned}\tag{1}$$

where v and w are the displacement components of the neutral axis of the beam along the y- and z-axes, respectively; ρ and A are the mass density and the cross-sectional area of the beam, respectively; c and E are the viscous damping coefficient and Young's modulus of the beam, respectively; I_{yy}, I_{zz} and I_{yz} are the moments and product of area of the beam, respectively. Note that I_{yy}, I_{zz} and I_{yz} are related to the moments of area about two principal axes $I_{y'y'}$ and $I_{z'z'}$ at every cross-section by

$$\begin{aligned}
I_{yy}&=I_{y'y'}\cos^2\gamma x/L+I_{z'z'}\sin^2\gamma x/L,\\
I_{zz}&=I_{z'z'}\cos^2\gamma x/L+I_{y'y'}\sin^2\gamma x/L,\\
I_{yz}&=\tfrac{1}{2}(I_{z'z'}-I_{y'y'})\sin 2\gamma x/L.
\end{aligned}\tag{2}$$

Boundary conditions

$$\begin{aligned}
&\text{At } x = 0: \quad v = w = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0. \\
&\text{At } x = L: \quad I_{zz}\frac{\partial^2 v}{\partial x^2} + I_{yz}\frac{\partial^2 w}{\partial x^2} = 0, \quad I_{yz}\frac{\partial^2 v}{\partial x^2} + I_{yy}\frac{\partial^2 w}{\partial x^2} = 0, \\
&E\frac{\partial}{\partial x}\left(I_{zz}\frac{\partial^2 v}{\partial x^2} + I_{yz}\frac{\partial^2 w}{\partial x^2}\right) + P\frac{\partial v}{\partial x} - \rho\left(I_{zz}\frac{\partial^3 v}{\partial t^2 \partial x} + I_{yz}\frac{\partial^3 w}{\partial t^2 \partial x}\right) = 0, \\
&E\frac{\partial}{\partial x}\left(I_{yz}\frac{\partial^2 v}{\partial x^2} + I_{yy}\frac{\partial^2 w}{\partial x^2}\right) + P\frac{\partial w}{\partial x} - \rho\left(I_{yz}\frac{\partial^3 v}{\partial t^2 \partial x} + I_{yy}\frac{\partial^3 w}{\partial t^2 \partial x}\right) = 0.
\end{aligned} \tag{3}$$

Eq. (1) is a set of coupled partial differential equations with time-dependent coefficients and cannot be solved directly. Therefore, the dependence on the spatial coordinate is first eliminated from Eq. (1). Due to the complexity in geometry, the eigen-solution of a pre-twisted beam cannot be found exactly (Rao, 1972). Consequently, an approximate method has to be used to eliminate the dependence on the spatial coordinate. In this work, the finite element method is adopted to obtain discretized system equations in time. Since these two equations of motion are coupled and include the fourth-order derivatives with respect to the spatial coordinate, the nodal variables should contain two nodal displacements (v, w) and two nodal slopes $(\partial v/\partial x, \partial w/\partial x)$. For a two-noded element, the displacement field within an element is interpolated by

$$v(x,t) = \sum_{j=1}^{4} d_j(t)\psi_j(x), \quad w(x,t) = \sum_{j=1}^{4} e_j(t)\psi_j(x), \tag{4}$$

where d_j and e_j are nodal parameters containing $(v, \partial v/\partial x)$ and $(w, \partial w/\partial x)$ at each node, respectively; $\psi_j(x)$ are the Hermite cubic interpolation functions. Substituting the displacement field into the equations of motion and going through the Galerkin procedure yields the discretized equation for the pre-twisted beam,

$$\mu^2[M]\ddot{\Delta} + 2\mu^2\left(\Omega[G] + \frac{c}{2\rho A}[C]\right)\dot{\Delta} + \left([K_e] + \frac{PL^2}{EI_0}[Q] + \mu^2\Omega^2[K_g] + \mu^2\dot{\Omega}[G]\right)\Delta = 0, \tag{5}$$

where $\mu^2 = \rho A L^4/EI_0$, and I_0 is the moment of area about the y-axis at the clamped end; $[M]$, $[G]$, $[C]$, and $[K_e]$ are the mass, gyroscopic, damping and elastic stiffness matrices, respectively; $[K_g]$ and $[Q]$ are geometric stiffness matrices due to spinning and the axial force, respectively; Δ is a column matrix formed by all nodal parameters, and a overdot denotes a differentiation with respect to time t. Note that the matrices $[M]$, $[C]$, $[K_e]$, $[K_g]$ and $[Q]$ are symmetric, while $[G]$ is skew-symmetric. Details of these matrices can be found in an earlier research work conducted by the authors (Young and Gau, 2003).

Eq. (5) is a set of the second-order ordinary differential equations with variable coefficients. To improve the solvability of Eq. (5), a modal analysis suitable for gyroscopic systems is applied to uncouple the undamped, autonomous terms in the equation. In this work, the spin rate of the beam is characterized as a small periodic perturbation $\Omega_1(t)$ superimposed on a constant speed Ω_0, and the axial force is assumed as the sum of a static force P_0 and a weakly stationary random process with a zero mean $P_1(t)$, i.e., $\Omega(t) = \Omega_0 + \Omega_1(t)$ and $P(t) = P_0 + P_1(t)$. Therefore, Eq. (6) can be rewritten into a set of the first-order differential equations in the non-dimensional form,

$$\begin{aligned}
\begin{bmatrix} [M] & [0] \\ [0] & [K_t] \end{bmatrix}\mathbf{p}' + \begin{bmatrix} 2\mu\Omega_0[G] & [K_t] \\ -[K_t] & [0] \end{bmatrix}\mathbf{p} = -\Bigg(& 2\alpha\mu\Omega_0\begin{bmatrix} [C] & [0] \\ [0] & [0] \end{bmatrix} + 2\mu\Omega_1(\tau)\begin{bmatrix} [G] & [0] \\ [0] & [0] \end{bmatrix} \\
& + \mu^2(2\Omega_0\Omega_1 + \Omega_1^2)\begin{bmatrix} [0] & [K_g] \\ [0] & [0] \end{bmatrix} + \mu\Omega_1'(\tau)\begin{bmatrix} [0] & [G] \\ [0] & [0] \end{bmatrix} \\
& + \frac{P_1(\tau)L^2}{EI_0}\begin{bmatrix} [0] & [Q] \\ [0] & [0] \end{bmatrix}\Bigg)\mathbf{p},
\end{aligned} \tag{6}$$

where $[K_t] = [K_e] + \mu^2\Omega_0^2[K_g] + (P_0L^2/EI_0)[Q]$, $\alpha = c/2\rho A\Omega_0$, $\tau = t/\mu$ and $\mathbf{p} = \begin{Bmatrix} \Delta' \\ \Delta \end{Bmatrix}$; a prime denotes a differentiation with respect to the dimensionless temporal variable τ . The eigenvalues of the corresponding undamped, autonomous system of Eq. (6), the system defined by the left-hand side of Eq. (6), appear in complex conjugate pairs, i.e., $s = \pm i\omega_n$, $n = 1, 2, \ldots, N$, where ω_n are the non-dimensional natural frequencies of the pre-twisted beam with a constant spin rate Ω_0, and N is the total degrees of freedom of the discretized system. The eigenvectors of the corresponding undamped, autonomous system of Eq. (6) also appear in complex conjugate pairs, i.e., $\mathbf{x} = \mathbf{y}_n \pm i\mathbf{z}_n$, $n = 1, 2, \ldots, N$.

Introduce a linear transformation $\mathbf{p} = [R]\varsigma$, where $[R]$ is the matrix formed by all normalized $\mathbf{y}_n$ and $\mathbf{z}_n$. Substituting this transformation into Eq. (6), pre-multiplying the transpose of $[R]$ and using the orthogonality of eigenvectors yield the following partially uncoupled equation,

$$\varsigma' + [\Lambda]\varsigma = -\left\{2\alpha[C^*] + 2\frac{\Omega_1}{\Omega_0}[G^*] + \left(2\frac{\Omega_1}{\Omega_0} + \frac{\Omega_1^2}{\Omega_0^2}\right)[K^*] + \frac{\Omega_1'}{\Omega_0}[H^*] + \frac{P_1}{P_0}[Q^*]\right\}\varsigma, \tag{7}$$

where $[\Lambda]$ is a block-diagonal matrix of the form

$$[\Lambda] = \text{block-diagonal}\begin{bmatrix} 0 & -\omega_n \\ \omega_n & 0 \end{bmatrix}, \quad [C^*] = \mu\Omega_0[R]^{\mathrm{T}}\begin{bmatrix} [C] & [0] \\ [0] & [0] \end{bmatrix}[R],$$

$$[G^*] = \mu\Omega_0[R]^{\mathrm{T}}\begin{bmatrix} [G] & [0] \\ [0] & [0] \end{bmatrix}[R], \quad [K^*] = \mu^2\Omega_0^2[R]^{\mathrm{T}}\begin{bmatrix} [0] & [K_g] \\ [0] & [0] \end{bmatrix}[R],$$

$$[H^*] = \mu\Omega_0[R]^{\mathrm{T}}\begin{bmatrix} [0] & [G] \\ [0] & [0] \end{bmatrix}[R], \quad [Q^*] = \frac{P_0L^2}{EI_0}[R]^{\mathrm{T}}\begin{bmatrix} [0] & [Q] \\ [0] & [0] \end{bmatrix}[R].$$

Again $[C^*]$ is symmetric, and $[G^*]$ is skew-symmetric due to the property of the congruent transformation. The terms on the left-hand side of Eq. (7) are uncoupled in a block-wise sense; however, those on the right-hand side of the equation are still coupled together. To match the form of $[\Lambda]$, the matrices on the right-hand side of the equation are partitioned into N^2 blocks of 2×2 matrices. Hence, Eq. (7) can be rewritten into the following form,

$$\begin{aligned} \xi_n' - \omega_n\eta_n = {} & -2\alpha\sum_{j=1}^{N}(c_{nj}^{11}\xi_j + c_{nj}^{12}\eta_j) - 2\frac{\Omega_1}{\Omega_0}\sum_{j=1}^{N}(g_{nj}^{11}\xi_j + g_{nj}^{12}\eta_j) - \left(2\frac{\Omega_1}{\Omega_0} + \frac{\Omega_1^2}{\Omega_0^2}\right)\sum_{j=1}^{N}(k_{nj}^{11}\xi_j + k_{nj}^{12}\eta_j) \\ & - \frac{\Omega_1'}{\Omega_0}\sum_{j=1}^{N}(h_{nj}^{11}\xi_j + h_{nj}^{12}\eta_j) - \frac{P_1}{P_0}\sum_{j=1}^{N}(q_{nj}^{11}\xi_j + q_{nj}^{12}\eta_j), \end{aligned}$$

$$\begin{aligned} \eta_n' + \omega_n\xi_n = {} & -2\alpha\sum_{j=1}^{N}(c_{nj}^{21}\xi_j + c_{nj}^{22}\eta_j) - 2\frac{\Omega_1}{\Omega_0}\sum_{j=1}^{N}(g_{nj}^{21}\xi_j + g_{nj}^{22}\eta_j) - \left(2\frac{\Omega_1}{\Omega_0} + \frac{\Omega_1^2}{\Omega_0^2}\right)\sum_{j=1}^{N}(k_{nj}^{21}\xi_j + k_{nj}^{22}\eta_j) \\ & - \Omega_1'\sum_{j=1}^{N}(h_{nj}^{21}\xi_j + h_{nj}^{22}\eta_j) - \frac{P_1}{P_0}\sum_{j=1}^{N}(q_{nj}^{21}\xi_j + q_{nj}^{22}\eta_j) \quad n = 1, 2, \ldots, N, \end{aligned} \tag{8}$$

where ξ_n and η_n are the $(2n-1)$th and the $2n$th entries of ς; c_{nj}^{im}, g_{nj}^{im}, k_{nj}^{im}, h_{nj}^{im} and q_{nj}^{im} are the i–mth entries of the n–jth blocks of $[C^*]$, $[G^*]$, $[K^*]$, $[H^*]$ and $[Q^*]$, respectively.

3. The method of stochastic averaging

The spin rate perturbation $\Omega_1(\tau)$ is assumed to be periodic and small compared to Ω_0; therefore, Ω_1/Ω_0 can be expanded into a Fourier series of the form (Young, 1991),

$$\frac{\Omega_1(\tau)}{\Omega_0} = \varepsilon \sum_{m=1}^{M} (f_m e^{i\beta_m \tau} + \bar{f}_m e^{-i\beta_m \tau}), \tag{9}$$

with $\beta_m = m\beta$, where β is called the perturbation frequency; ε is a small parameter; $\bar{f}_m$ is the complex conjugate of f_m. Assume the solutions of Eq. (8) to be of the form,

$$\xi_n = z_n e^{i\kappa_n \beta_s \tau} + \bar{z}_n e^{-i\kappa_n \beta_s \tau}; \quad \eta_n = i(z_n e^{i\kappa_n \beta_s \tau} - \bar{z}_n e^{-i\kappa_n \beta_s \tau}), \ \ n = 1, 2, \ldots, N, \tag{10}$$

where $\kappa_n = \omega_n(1 - \varepsilon\lambda)/\beta_s$, in which λ is a detuning parameter. Substituting Eqs. (9) and (10) into Eq. (8) yields

$$\begin{aligned}
\frac{dz_n}{d\tau} = \varepsilon \Bigg\{ & i\omega_n \lambda z_n - \tilde{\alpha} \sum_{r=1}^{N} [(c_{nr}^{11} + c_{nr}^{22} + ic_{nr}^{12} - ic_{nr}^{21}) z_r e^{i(\kappa_r - \kappa_n)\beta_s \tau} + (c_{nr}^{11} - c_{nr}^{22} - ic_{nr}^{12} - ic_{nr}^{21}) \bar{z}_r e^{-i(\kappa_r + \kappa_n)\beta_s \tau}] \\
& - \sum_{m=1}^{M} \sum_{r=1}^{N} [f_m (g_{nr}^{11} + g_{nr}^{22} + ig_{nr}^{12} - ig_{nr}^{21}) z_r e^{i(\delta_m + \kappa_r - \kappa_n)\beta_s \tau} + f_m (g_{nr}^{11} - g_{nr}^{22} - ig_{nr}^{12} - ig_{nr}^{21}) \bar{z}_r e^{i(\delta_m - \kappa_r - \kappa_n)\beta_s \tau} \\
& + \bar{f}_m (g_{nr}^{11} + g_{nr}^{22} + ig_{nr}^{12} - ig_{nr}^{21}) z_r e^{-i(\delta_m - \kappa_r + \kappa_n)\beta_s \tau} + \bar{f}_m (g_{nr}^{11} - g_{nr}^{22} - ig_{nr}^{12} - ig_{nr}^{21}) \bar{z}_r e^{-i(\delta_m + \kappa_r + \kappa_n)\beta_s \tau}] \\
& - \frac{1}{2} \sum_{m=1}^{M} \sum_{r=1}^{N} \beta_m [f_m (ih_{nr}^{11} + ih_{nr}^{22} - h_{nr}^{12} + h_{nr}^{21}) z_r e^{i(\delta_m + \kappa_r - \kappa_n)\beta_s \tau} + f_m (ih_{nr}^{11} - ih_{nr}^{22} + h_{nr}^{12} + h_{nr}^{21}) \bar{z}_r e^{i(\delta_m - \kappa_r - \kappa_n)\beta_s \tau} \\
& + \bar{f}_m (ih_{nr}^{11} + ih_{nr}^{22} - h_{nr}^{12} + h_{nr}^{21}) z_r e^{-i(\delta_m - \kappa_r + \kappa_n)\beta_s \tau} - \bar{f}_m (ih_{nr}^{11} - ih_{nr}^{22} + h_{nr}^{12} + h_{nr}^{21}) \bar{z}_r e^{-i(\delta_m + \kappa_r + \kappa_n)\beta_s \tau}] \\
& - \sum_{m=1}^{M} \sum_{r=1}^{N} \Big[f_m (k_{nr}^{11} + k_{nr}^{22} + ik_{nr}^{12} - ik_{nr}^{21}) z_r e^{i(\delta_m + \kappa_r - \kappa_n)\beta_s \tau} + f_m (k_{nr}^{11} - k_{nr}^{22} - ik_{nr}^{12} - ik_{nr}^{21}) \bar{z}_r e^{i(\delta_m - \kappa_r - \kappa_n)\beta_s \tau} \\
& + \bar{f}_m (k_{nr}^{11} + k_{nr}^{22} + ik_{nr}^{12} - ik_{nr}^{21}) z_r e^{-i(\delta_m - \kappa_r + \kappa_n)\beta_s \tau} + \bar{f}_m (k_{nr}^{11} - k_{nr}^{22} - ik_{nr}^{12} - ik_{nr}^{21}) \bar{z}_r e^{-i(\delta_m + \kappa_r + \kappa_n)\beta_s \tau} \Big] \Bigg\} \\
& - \frac{\sqrt{\varepsilon}}{2} \zeta(\tau) \sum_{r=1}^{N} \Big[(q_{nr}^{11} + q_{nr}^{22} + iq_{nr}^{12} - iq_{nr}^{21}) z_r e^{i(\kappa_r - \kappa_n)\beta_s \tau} + (q_{nr}^{11} - q_{nr}^{22} - iq_{nr}^{12} - iq_{nr}^{21}) \bar{z}_r e^{-i(\kappa_r + \kappa_n)\beta_s \tau} \Big] \\
& + O(\varepsilon^2) \quad n = 1, 2, \ldots, N,
\end{aligned} \tag{11}$$

where $\delta_m = \beta_m/\beta_s$; $\alpha = \varepsilon\tilde{\alpha}$ and $P_1(\tau)/P_0 = \varepsilon^{1/2}\zeta(\tau)$ are assumed in order that the contributions of the damping, periodic and random excitations to the system response are commensurable. The corresponding equation for $d\bar{z}_n/d\tau$ can be obtained by conjugating the above equation for $dz_n/d\tau$.

To a first approximation, $z_n(\tau)$ may be replaced by the solutions of the time-averaged equations by using the method of stochastic averaging (Khas'minskii, 1966). Upon applying the averaging procedure, there exist three possible resonant frequency combinations of β_s, ω_p and ω_q to this order of approximation. All three cases of frequency combinations will be investigated separately in the following.

3.1. The case of $\beta_s \approx 2\omega_p$

In this case, $\delta_s = 2\kappa_p$, and the averaged equations take the forms,

$$\begin{aligned} \mathrm{d}z_n = \varepsilon\Bigg\{ & \mathrm{i}\omega_n\lambda z_n - \tilde{\alpha}(c_{nn}^{11}+c_{nn}^{22})z_n + \frac{1}{8}\sum_{r=1}^{N}[(q_{nr}^{11}+q_{nr}^{22}+\mathrm{i}q_{nr}^{12}-\mathrm{i}q_{nr}^{21})(q_{rn}^{11}+q_{rn}^{22}+\mathrm{i}q_{rn}^{12}-\mathrm{i}q_{rn}^{21})z_n(S((\kappa_n-\kappa_r)\beta_s) \\ & -\mathrm{i}\psi((\kappa_n-\kappa_r)\beta_s)) + (q_{nr}^{11}-q_{nr}^{22}-\mathrm{i}q_{nr}^{12}-\mathrm{i}q_{nr}^{21})(q_{rn}^{11}-q_{rn}^{22}+\mathrm{i}q_{rn}^{12}+\mathrm{i}q_{rn}^{21})z_n(S((\kappa_n+\kappa_r)\beta_s) \\ & -\mathrm{i}\psi((\kappa_n+\kappa_r)\beta_s))]\Bigg\}\mathrm{d}\tau + \sum_{r=1}^{2N}\sigma_{nr}\mathrm{d}B_r \quad n=1,2,\ldots,N;\ n\neq p, \end{aligned} \tag{12}$$

$$\begin{aligned} \mathrm{d}z_p = \varepsilon\Bigg\{ & \mathrm{i}\omega_p\lambda z_p - \tilde{\alpha}(c_{pp}^{11}+c_{pp}^{22})z_p - f_s(g_{pp}^{11}-g_{pp}^{22}-\mathrm{i}g_{pp}^{12}-\mathrm{i}g_{pp}^{21})\bar{z}_p - \frac{1}{2}f_s\beta_s(\mathrm{i}h_{pp}^{11}-\mathrm{i}h_{pp}^{22}+h_{pp}^{12}+h_{pp}^{21})\bar{z}_p \\ & - f_s(k_{pp}^{11}-k_{pp}^{22}-\mathrm{i}k_{pp}^{12}-\mathrm{i}k_{pp}^{21})\bar{z}_p + \frac{1}{8}\sum_{r=1}^{N}\Big[(q_{pr}^{11}+q_{pr}^{22}+\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{rp}^{11}+q_{rp}^{22}+\mathrm{i}q_{rp}^{12}-\mathrm{i}q_{rp}^{21}) \\ & \times z_p(S((\kappa_p-\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p-\kappa_r)\beta_s)) + (q_{pr}^{11}-q_{pr}^{22}-\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{rp}^{11}-q_{rp}^{22}+\mathrm{i}q_{rp}^{12}+\mathrm{i}q_{rp}^{21}) \\ & \times z_p(S((\kappa_p+\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p+\kappa_r)\beta_s))\Big]\Bigg\}\mathrm{d}\tau + \sum_{r=1}^{2N}\sigma_{pr}\mathrm{d}B_r, \end{aligned} \tag{13}$$

where $S(\omega) = 2\int_0^\infty R(\tau)\cos\omega\tau\,\mathrm{d}\tau$ and $\psi(\omega) = 2\int_0^\infty R(\tau)\sin\omega\tau\,\mathrm{d}\tau$, in which $R(\tau)$ is the autocorrelation function of $\zeta(\tau)$; B_r are mutually independent unit Wiener processes; σ_{nr} are elements of the diffusion matrix $[\sigma]$. Detail expressions for $[\sigma\sigma^{\mathrm{T}}]$ are given in Appendix A. Note that in Eqs. (12) and (13), $c_{nn}^{12} = c_{nn}^{21}$ and $c_{pp}^{12} = c_{pp}^{21}$ are used since the matrix $[C]$ is symmetric. The corresponding equation for $\bar{z}_p$ can be obtained by conjugating the above equation for z_p.

3.2. The case of $\beta_s \approx \omega_p + \omega_q$

In this case, $\delta_s = \kappa_p + \kappa_q$, and the averaged equations take the forms,

$$\begin{aligned} \mathrm{d}z_p = \varepsilon\Bigg\{ & \mathrm{i}\omega_p\lambda z_p - \tilde{\alpha}(c_{pp}^{11}+c_{pp}^{22})z_p - f_s(g_{pq}^{11}-g_{pq}^{22}-\mathrm{i}g_{pq}^{12}-\mathrm{i}g_{pq}^{21})\bar{z}_q - \frac{1}{2}f_s\beta_s(\mathrm{i}h_{pq}^{11}-\mathrm{i}h_{pq}^{22}+h_{pq}^{12}+h_{pq}^{21})\bar{z}_q \\ & - f_s(k_{pq}^{11}-k_{pq}^{22}-\mathrm{i}k_{pq}^{12}-\mathrm{i}k_{pq}^{21})\bar{z}_q + \frac{1}{8}\sum_{r=1}^{N}\Big[(q_{pr}^{11}+q_{pr}^{22}+\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{rp}^{11}+q_{rp}^{22}+\mathrm{i}q_{rp}^{12}-\mathrm{i}q_{rp}^{21}) \\ & \times z_p(S((\kappa_p-\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p-\kappa_r)\beta_s)) + (q_{pr}^{11}-q_{pr}^{22}-\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{rp}^{11}-q_{rp}^{22}+\mathrm{i}q_{rp}^{12}+\mathrm{i}q_{rp}^{21}) \\ & \times z_p(S((\kappa_p+\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p+\kappa_r)\beta_s))\Big]\Bigg\}\mathrm{d}\tau + \sum_{r=1}^{2N}\sigma_{pr}\mathrm{d}B_r, \end{aligned} \tag{14}$$

$$\begin{aligned} \mathrm{d}\bar{z}_q = \varepsilon\Bigg\{ & -\mathrm{i}\omega_q\lambda\bar{z}_q - \tilde{\alpha}(c_{qq}^{11}+c_{qq}^{22})\bar{z}_q - \bar{f}_s(g_{qp}^{11}-g_{qp}^{22}+\mathrm{i}g_{qp}^{12}+\mathrm{i}g_{qp}^{21})z_p - \frac{1}{2}\bar{f}_s\beta_s(-\mathrm{i}h_{qp}^{11}+\mathrm{i}h_{qp}^{22}+h_{qp}^{12}+h_{qp}^{21})z_p \\ & - \bar{f}_s(k_{qp}^{11}-k_{qp}^{22}+\mathrm{i}k_{qp}^{12}+\mathrm{i}k_{qp}^{21})z_p + \frac{1}{8}\sum_{r=1}^{N}[(q_{qr}^{11}+q_{qr}^{22}-\mathrm{i}q_{qr}^{12}+\mathrm{i}q_{qr}^{21})(q_{rq}^{11}+q_{rq}^{22}-\mathrm{i}q_{rq}^{12}+\mathrm{i}q_{rq}^{21}) \\ & \times \bar{z}_q(S((\kappa_q-\kappa_r)\beta_s)+\mathrm{i}\psi((\kappa_q-\kappa_r)\beta_s)) + (q_{qr}^{11}-q_{qr}^{22}+\mathrm{i}q_{qr}^{12}+\mathrm{i}q_{qr}^{21})(q_{rq}^{11}-q_{rq}^{22}-\mathrm{i}q_{rq}^{12}-\mathrm{i}q_{rq}^{21}) \\ & \times \bar{z}_q(S((\kappa_q+\kappa_r)\beta_s)+\mathrm{i}\psi((\kappa_q+\kappa_r)\beta_s))]\Bigg\}\mathrm{d}\tau + \sum_{r=1}^{2N}\sigma_{(q+N),r}\mathrm{d}B_r. \end{aligned} \tag{15}$$

The equations for z_n, $n \neq p$ or q, are the same as Eq. (12).

3.3. The case of $\beta_s \approx \omega_p - \omega_q$

In this case, $\delta_s = \kappa_p - \kappa_q$, and the averaged equations take the forms,

$$\begin{aligned}
\mathrm{d}z_p = \varepsilon\Bigg\{ & \mathrm{i}\omega_p\lambda z_p - \tilde{\alpha}(c_{pp}^{11} + c_{pp}^{22})z_p - f_s(g_{pq}^{11} + g_{pq}^{22} + \mathrm{i}g_{pq}^{12} - \mathrm{i}g_{pq}^{21})z_q - \frac{1}{2}f_s\beta_s(\mathrm{i}h_{pq}^{11} + \mathrm{i}h_{pq}^{22} - h_{pq}^{12} + h_{pq}^{21})z_q \\
& - f_s(k_{pq}^{11} + k_{pq}^{22} + \mathrm{i}k_{pq}^{12} - \mathrm{i}k_{pq}^{21})z_q + \frac{1}{8}\sum_{r=1}^{N}\Big[(q_{pr}^{11} + q_{pr}^{22} + \mathrm{i}q_{pr}^{12} - \mathrm{i}q_{pr}^{21})(q_{rp}^{11} + q_{rp}^{22} + \mathrm{i}q_{rp}^{12} - \mathrm{i}q_{rp}^{21}) \\
& \times z_p(S((\kappa_p - \kappa_r)\beta_s) - \mathrm{i}\psi((\kappa_p - \kappa_r)\beta_s)) + (q_{pr}^{11} - q_{pr}^{22} - \mathrm{i}q_{pr}^{12} - \mathrm{i}q_{pr}^{21})(q_{rp}^{11} - q_{rp}^{22} + \mathrm{i}q_{rp}^{12} + \mathrm{i}q_{rp}^{21}) \\
& \times z_p(S((\kappa_p + \kappa_r)\beta_s) - \mathrm{i}\psi((\kappa_p + \kappa_r)\beta_s))\Big]\Bigg\}\mathrm{d}\tau + \sum_{r=1}^{2N}\sigma_{pr}\mathrm{d}B_r,
\end{aligned} \tag{16}$$

$$\begin{aligned}
\mathrm{d}z_q = \varepsilon\Bigg\{ & \mathrm{i}\omega_q\lambda z_q - \tilde{\alpha}(c_{qq}^{11} + c_{qq}^{22})z_q - \bar{f}_s(g_{qp}^{11} + g_{qp}^{22} + \mathrm{i}g_{qp}^{12} - \mathrm{i}g_{qp}^{21})z_p + \frac{1}{2}\bar{f}_s\beta_s(\mathrm{i}h_{qp}^{11} + \mathrm{i}h_{qp}^{22} - h_{qp}^{12} + h_{qp}^{21})z_p \\
& - \bar{f}_s(k_{qp}^{11} + k_{qp}^{22} + \mathrm{i}k_{qp}^{12} - \mathrm{i}k_{qp}^{21})z_p + \frac{1}{8}\sum_{r=1}^{N}\Big[(q_{qr}^{11} + q_{qr}^{22} + \mathrm{i}q_{qr}^{12} - \mathrm{i}q_{qr}^{21})(q_{rq}^{11} + q_{rq}^{22} + \mathrm{i}q_{rq}^{12} - \mathrm{i}q_{rq}^{21}) \\
& \times z_q(S((\kappa_q - \kappa_r)\beta_s) - \mathrm{i}\psi((\kappa_q - \kappa_r)\beta_s)) + (q_{qr}^{11} - q_{qr}^{22} - \mathrm{i}q_{qr}^{12} - \mathrm{i}q_{qr}^{21})(q_{rq}^{11} - q_{rq}^{22} + \mathrm{i}q_{rq}^{12} + \mathrm{i}q_{rq}^{21}) \\
& \times z_q(S((\kappa_q + \kappa_r)\beta_s) - \mathrm{i}\psi((\kappa_q + \kappa_r)\beta_s))\Big]\Bigg\}\mathrm{d}\tau + \sum_{r=1}^{2N}\sigma_{qr}\mathrm{d}B_r.
\end{aligned} \tag{17}$$

The equations for z_n, $n \neq p$ or q, are the same as Eq. (12).

4. Stability analysis

In this work, the first- and second-moment stability of the pre-twisted beam is considered. All three cases of resonant frequency combinations will be investigated separately in the following.

4.1. The case of $\beta_s \approx 2\omega_p$

The differential equations governing the first-moment of the system response are obtained by taking the expectation on both sides of Eqs. (12) and (13). It is evident that the resulting equations will be the same as Eqs. (12) and (13) with the stochastic terms absent and the response variables z_n and $\bar{z}_n$ replaced by their expectations, and the first-moment equations can be written as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}E[z_n] = a_n E[z_n] \quad n = 1, 2, \ldots, N; \;\; n \neq p, \\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} E[z_p] \\ E[\bar{z}_p] \end{bmatrix} = [A_1]\begin{bmatrix} E[z_p] \\ E[\bar{z}_p] \end{bmatrix}.
\end{aligned} \tag{18}$$

Note that the equations for $E[z_n]$ are mutually independent when $n \neq p$, but the equation for $E[z_p]$ is coupled with the equation for $E[\bar{z}_p]$. The first-moment stability of the system is assured if all the expectations of the response variables are bounded, i.e., the first-moment stability of the system can be secured if every a_n is negative and if the real parts of the eigenvalues of $[A_1]$ are negative. It is found that the latter is stricter than the former. Consequently, the first-moment stability criterion in this case corresponds to the one that renders the real part of an eigenvalue of $[A_1]$ being zero.

By using Ito's differential rule, the equations for $d(z_n\bar{z}_n)$, $d(z_p\bar{z}_p)$ and $d(z_p^2)$ can be obtained as follows:

$$\mathrm{d}(z_n\bar{z}_n) = U_n(z_1\bar{z}_1,\ldots,z_N\bar{z}_N)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{nr}\bar{z}_n\mathrm{d}B_r + \sigma_{(n+N),r}z_n\mathrm{d}B_r), \quad n \neq p, \tag{19a}$$

$$\mathrm{d}(z_p\bar{z}_p) = U_{p1}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_p^2,\bar{z}_p^2)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{pr}\bar{z}_p\mathrm{d}B_r + \sigma_{(p+N),r}z_p\mathrm{d}B_r), \tag{19b}$$

$$\mathrm{d}(z_p^2) = U_{p2}(z_p\bar{z}_p,z_p^2)\mathrm{d}\tau + \sum_{r=1}^{2N}2\sigma_{pr}z_p\mathrm{d}B_r, \tag{19c}$$

where U_n, U_{p1} and U_{p2} are functions given in Appendix A. The equation for $\bar{z}_p^2$ can be found by conjugating Eq. (19c). The differential equations governing the second-moment of the system response are obtained by taking the expectation on both sides of Eqs. (19a)–(19c) and the conjugation of Eq. (19c). It is evident that the resulting equations will be the same as Eqs. (19a)–(19c) and the conjugation of Eq. (19c) with the stochastic terms absent and the response variables $z_n\bar{z}_n$, $z_p\bar{z}_p$, z_p^2 and $\bar{z}_p^2$ replaced by their expectations, and the second-moment equations can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{Z}_m = [A_2]\mathbf{Z}_m, \tag{20}$$

where $\mathbf{Z}_m = [E[z_1\bar{z}_1],E[z_2\bar{z}_2],\ldots,E[z_N\bar{z}_N],E[z_p^2],E[\bar{z}_p^2]]^{\mathrm{T}}$. The second-moment stability of the system is assured if $\mathbf{Z}_m$ is bounded, i.e., the second-moment stability of the system can be secured if the real parts of all eigenvalues of $[A_2]$ are negative. Consequently, the second-moment stability criterion in this case corresponds to the one that renders the real part of an eigenvalue of $[A_2]$ being zero.

4.2. The case of $\beta_s \approx \omega_p + \omega_q$

Taking the expectation on both sides of Eqs. (12), (14) and (15) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}E[z_n] = a_nE[z_n] \quad n = 1,2,\ldots,N;\ n \neq p,q,\\ &\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}E[z_p]\\E[\bar{z}_q]\end{bmatrix} = [S_1]\begin{bmatrix}E[z_p]\\E[\bar{z}_q]\end{bmatrix}.\end{aligned} \tag{21}$$

Note that the equations for $E[z_n]$ are mutually independent when $n \neq p$ and q, but the equation for $E[z_p]$ is coupled with the equation for $E[\bar{z}_q]$. The first-moment stability of the system can be assured if every a_n is negative and if the real parts of the eigenvalues of $[S_1]$ are negative. It is found that the latter is more severe than the former. Consequently, the first-moment stability criterion in this case corresponds to the one that renders the real part of an eigenvalue of $[S_1]$ being zero.

By using Ito's differential rule, the equations for $\mathrm{d}(z_p\bar{z}_p)$, $\mathrm{d}(z_q\bar{z}_q)$ and $\mathrm{d}(z_p\bar{z}_q)$ can be obtained as follows:

$$\mathrm{d}(z_p\bar{z}_p) = U_{s1}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_pz_q,\bar{z}_p\bar{z}_q)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{pr}\bar{z}_p\mathrm{d}B_r + \sigma_{(p+N),r}z_p\mathrm{d}B_r), \tag{22a}$$

$$\mathrm{d}(z_q\bar{z}_q) = U_{s2}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_pz_q,\bar{z}_p\bar{z}_q)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{qr}\bar{z}_q\mathrm{d}B_r + \sigma_{(q+N),r}z_q\mathrm{d}B_r), \tag{22b}$$

$$\mathrm{d}(z_pz_q) = U_{s3}(z_p\bar{z}_p,z_q\bar{z}_q,z_pz_q)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{pr}z_q\mathrm{d}B_r + \sigma_{qr}z_p\mathrm{d}B_r), \tag{22c}$$

where U_{s1}, U_{s2} and U_{s3} are functions given in Appendix A. The equation for $\bar{z}_p\bar{z}_q$ can be found by conjugating Eq. (22c). Taking the expectation on both sides of Eqs. (19a), (22a)–(22c) and the conjugation of Eq. (22c) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{Z}_s = [S_2]\mathbf{Z}_s, \tag{23}$$

where $\mathbf{Z}_s = [E[z_1\bar{z}_1],E[z_2\bar{z}_2],\ldots,E[z_N\bar{z}_N],E[z_pz_q],E[\bar{z}_p\bar{z}_q]]^{\mathrm{T}}$. The second-moment stability of the system can be assured if the real parts of all eigenvalues of $[S_2]$ are negative. Consequently, the second-moment stability criterion in this case corresponds to the one that renders the real part of an eigenvalue of $[S_2]$ being zero.

4.3. The case of $\beta_s \approx \omega_p - \omega_q$

Taking the expectation on both sides of Eqs. (12), (16) and (17) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}E[z_n] = a_nE[z_n] \quad n = 1,2,\ldots,N;\ \ n \neq p,q,\\ &\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}E[z_p]\\ E[z_q]\end{bmatrix} = [D_1]\begin{bmatrix}E[z_p]\\ E[z_q]\end{bmatrix}.\end{aligned} \tag{24}$$

Again the equations for $E[z_n]$ are mutually independent when $n \neq p$ and q, but the equation for $E[z_p]$ is coupled with the equation for $E[z_q]$. The first-moment stability of the system can be assured if every a_n is negative and if the real parts of the eigenvalues of $[D_1]$ are negative. It is found that the latter is stricter than the former. Consequently, the first-moment stability criterion in this case corresponds to the one that renders the real part of an eigenvalue of $[D_1]$ being zero.

By using Ito's differential rule, the equations for $\mathrm{d}(z_p\bar{z}_p)$, $\mathrm{d}(z_q\bar{z}_q)$ and $\mathrm{d}(z_p\bar{z}_q)$ can be obtained as follows:

$$\mathrm{d}(z_p\bar{z}_p) = U_{d1}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_p\bar{z}_q,\bar{z}_pz_q)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{pr}\bar{z}_p\mathrm{d}B_r + \sigma_{(p+N),r}z_p\mathrm{d}B_r), \tag{25a}$$

$$\mathrm{d}(z_q\bar{z}_q) = U_{d2}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_p\bar{z}_q,\bar{z}_pz_q)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{qr}\bar{z}_q\mathrm{d}B_r + \sigma_{(q+N),r}z_q\mathrm{d}B_r), \tag{25b}$$

$$\mathrm{d}(z_p\bar{z}_q) = U_{d3}(z_p\bar{z}_p,z_q\bar{z}_q,z_p\bar{z}_q)\mathrm{d}\tau + \sum_{r=1}^{2N}(\sigma_{qr}\bar{z}_p\mathrm{d}B_r + \sigma_{(p+N),r}z_q\mathrm{d}B_r), \tag{25c}$$

where U_{d1}, U_{d2} and U_{d3} are functions given in Appendix A. The equation for $\bar{z}_pz_q$ can be found by conjugating Eq. (25c). Taking the expectation on both sides of Eqs. (19a), (25a)–(25c) and the conjugation of Eq. (25c) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{Z}_d = [D_2]\mathbf{Z}_d, \tag{26}$$

where $\mathbf{Z}_d = [E[z_1\bar{z}_1], E[z_2\bar{z}_2], \ldots, E[z_N\bar{z}_N], E[z_p\bar{z}_q], E[\bar{z}_p z_q]]^{\mathrm{T}}$. The second-moment stability of the system can be assured if the real parts of all eigenvalues of $[D_2]$ are negative. Consequently, the second-moment stability criterion in this case corresponds to the one that renders the real part of an eigenvalue of $[D_2]$ being zero.

5. Comparison with known results

When the axial force is static as the problem considered by Liao and Huang (1995b), i.e., the random part is absent, the beam is subjected to a parametric excitation only due to the periodically varying spin rate. In this situation, when the excitation frequency β_s is near $\omega_p + \omega_q$, the first-moment stability criterion is still the one that renders the real part of an eigenvalue of $[S_1]$ in Eq. (21) being zero. That gives

$$\lambda = \pm \frac{c_{pp}^{11} + c_{pp}^{22} + c_{qq}^{11} + c_{qq}^{22}}{\omega_p + \omega_q} \sqrt{\frac{\Lambda_{pq}\bar{\Lambda}_{qp}}{c_{pp}^{11} + c_{pp}^{22} + c_{qq}^{11} + c_{qq}^{22}} - \tilde{\alpha}^2}, \tag{27}$$

where

$$\Lambda_{pq} = f_s\left\{\left[g_{pq}^{11} - g_{pq}^{22} + \frac{1}{2}\beta_s(h_{pq}^{12} + h_{pq}^{21}) + k_{pq}^{11} - k_{pq}^{22}\right] + \mathrm{i}\left[-g_{pq}^{12} - g_{pq}^{21} + \frac{1}{2}\beta_s(h_{pq}^{11} - h_{pq}^{22}) - k_{pq}^{12} - k_{pq}^{21}\right]\right\}$$

and

$$\bar{\Lambda}_{qp} = f_s\left\{\left[g_{qp}^{11} - g_{qp}^{22} + \frac{1}{2}\beta_s(h_{qp}^{12} + h_{qp}^{21}) + k_{qp}^{11} - k_{qp}^{22}\right] + \mathrm{i}\left[g_{qp}^{12} + g_{qp}^{21} + \frac{1}{2}\beta_s(-h_{qp}^{11} + h_{qp}^{22}) + k_{qp}^{12} + k_{qp}^{21}\right]\right\}.$$

The detuning parameter λ in Eq. (27) is the same as that obtained by Liao and Huang (1995b). For the second-moment response, Eq. (23) can be divided into two parts: the equations for $z_n\bar{z}_n$, $n \neq p$ or q, which are mutually independent, and the remaining four equations for $z_p\bar{z}_p$, $z_q\bar{z}_q$, $z_p z_q$ and $\bar{z}_p\bar{z}_q$, which are still coupled in the absence of the random excitation. The second-moment response $z_n\bar{z}_n$ for $n \neq p$ or q decays exponentially, and the second-moment stability of the system is determined by the boundedness of $z_p\bar{z}_p$, $z_q\bar{z}_q$, $z_p z_q$ and $\bar{z}_p\bar{z}_q$, or equivalently by the real part of an eigenvalue of the coefficient matrix of the equations for $z_p\bar{z}_p$, $z_q\bar{z}_q$, $z_p z_q$ and $\bar{z}_p\bar{z}_q$ being zero. That gives the same detuning parameter λ shown in Eq. (27). Therefore, in the absence of the random excitation, the first-moment and the second-moment stability criteria become identical and are the same as that previously obtained by Liao and Huang (1995b). The case that β_s is near $2\omega_p$ can be obtained from the case that β_s is near $\omega_p + \omega_q$ by letting q equal to p.

When the excitation frequency β_s is near $\omega_p - \omega_q$, the first-moment stability criterion is still the one that renders the real part of an eigenvalue of $[D_1]$ in Eq. (24) being zero. That gives

$$\lambda = \pm \frac{c_{pp}^{11} + c_{pp}^{22} + c_{qq}^{11} + c_{qq}^{22}}{\omega_p + \omega_q} \sqrt{\frac{\Lambda_{pq}\Lambda_{qp}}{c_{pp}^{11} + c_{pp}^{22} + c_{qq}^{11} + c_{qq}^{22}} - \tilde{\alpha}^2}, \tag{28}$$

where

$$\Lambda_{pq} = f_s\left\{\left[g_{pq}^{11} + g_{pq}^{22} + \frac{1}{2}\beta_s(-h_{pq}^{12} + h_{pq}^{21}) + k_{pq}^{11} + k_{pq}^{22}\right] + \mathrm{i}\left[g_{pq}^{12} - g_{pq}^{21} + \frac{1}{2}\beta_s(h_{pq}^{11} + h_{pq}^{22}) + k_{pq}^{12} - k_{pq}^{21}\right]\right\}$$

and

$$\Lambda_{qp} = f_s\left\{\left[g_{qp}^{11} + g_{qp}^{22} + \frac{1}{2}\beta_s(-h_{qp}^{12} + h_{qp}^{21}) + k_{qp}^{11} + k_{qp}^{22}\right] + \mathrm{i}\left[g_{qp}^{12} - g_{qp}^{21} + \frac{1}{2}\beta_s(h_{qp}^{11} + h_{qp}^{22}) + k_{qp}^{12} - k_{qp}^{21}\right]\right\}.$$

The detuning parameter λ in Eq. (28) is the same as that obtained by Liao and Huang (1995b). For the second-moment response, Eq. (26) can be divided into two parts: the equations for $z_n\bar{z}_n$, $n \neq p$ or q, which are mutually independent, and the remaining four equations for $z_p\bar{z}_p$, $z_q\bar{z}_q$, $z_p\bar{z}_q$ and $\bar{z}_pz_q$, which are still coupled in the absence of the random excitation. The second-moment response $z_n\bar{z}_n$ for $n \neq p$ or q decays exponentially, and the second-moment stability of the system is determined by the boundedness of $z_p\bar{z}_p$, $z_q\bar{z}_q$, $z_p\bar{z}_q$ and $\bar{z}_pz_q$, or equivalently by the real part of an eigenvalue of the coefficient matrix of these four second-moment equations being zero. That gives the same detuning parameter λ shown in Eq. (28). Again, in the absence of the random excitation, the first-moment and the second-moment stability criteria become identical and are the same as that previously obtained by Liao and Huang (1995b).

When the spin rate of the beam is constant as the problem considered previously by the authors (2001), the beam is acted upon by the random excitation only due to the axial force. In this situation, all the first-moment equations are mutually independent, and the first-moment equations for different resonant frequency combinations, Eqs. (18), (21) and (24), reduce to the same form with λ and f_s equal to 0. Therefore, the first-moment stability criterion is the boundedness of all the first-moment response $E[z_n]$, i.e., $a_n < 0$ for $n = 1, 2, \ldots, N$, and is irrelevant to the resonant frequency combinations.

The second-moment equations under different resonant frequency combinations, Eqs. (20), (23) and (26), can be divided into two parts: the equations for $z_n\bar{z}_n$, $n = 1, 2, \ldots, N$, which are still coupled in the absence of the parametric excitation, and the remaining two mutually independent equations for z_p^2 and $\bar{z}_p^2$ when β_s is close to $2\omega_p$, for z_pz_q and $\bar{z}_p\bar{z}_q$ when β_s is near $\omega_p + \omega_q$, and for $z_p\bar{z}_q$ and $\bar{z}_pz_q$ when β_s is near $\omega_p - \omega_q$. Since the equations for $z_n\bar{z}_n$ are self-complete, the remaining two equations for z_p^2 and $\bar{z}_p^2$, z_pz_q and $\bar{z}_p\bar{z}_q$, or $z_p\bar{z}_q$ and $\bar{z}_pz_q$ are redundant when considering the second-moment stability of the system. Therefore, the second-moment stability of the system is determined by the boundedness of all $z_n\bar{z}_n$, or equivalently by the real part of an eigenvalue of the coefficient matrix of the equations for $z_n\bar{z}_n$ being zero. Note that the equations for $z_n\bar{z}_n$ reduce to the same form for different resonant frequency combinations in the absence of the spin rate perturbation and are the same as that obtained previously by the authors (2001).

6. Numerical results and discussions

Before formally presenting the numerical results for the stability analysis, convergence studies of the finite element model have to be conducted first. According to the suggestion by the previous study (Liao and Dang, 1992), 25 uniform elements are used to model the spinning pre-twisted beam in this work. The numerical results for the lowest few natural frequencies agree excellently with those obtained by Liao and Dang (1992). As an application of the general solution, the spin rate perturbation is taken as $\Omega_1(t) = f\cos\beta t$, where f is assumed to be small compared to the average spin rate Ω_0. Hence, the small parameter ε is defined as $\varepsilon = f/\Omega_0$. In addition, the random process $P_1(t)/P_0$ is assumed as a Gaussian white noise with a spectral density S; therefore, $S(0) = S((\kappa_p \pm \kappa_q)\beta) = S$ and $\psi((\kappa_p \pm \kappa_q)\beta) = 0$. However, numerical results for more general periodic functions and other random processes can also be produced easily. Moreover, all the numerical results in this work are presented in non-dimensional forms; therefore, β, Ω_0 and P_0 in this section represent the non-dimensional expressions β/ω_0, Ω_0/ω_0 and PL^2/EI_0, respectively, where ω_0 is the fundamental natural frequency of a free, non-spinning prismatic beam, i.e., $\omega_0 = 3.5156\sqrt{EI_0/\rho AL^4}$.

Fig. 2 shows the effect of the viscous damping of the beam on the first-moment and the second-moment stability boundaries of a pre-twisted beam with a non-constant spin rate under an axial random force at the first main resonance $\beta \approx 2\omega_1$. In this figure, stability regions lie outside the stability boundaries. Note that, in this case, the beam is stable when acted upon by the axial random force only; therefore, the stability boundaries look like those subjected to the parametric excitation due to the spin rate perturbation only. It is observed that an increase in the viscous damping will reduce the unstable regions predicted by the first-moment and the second-moment stability criteria, respectively, and the effect on the second-moment sta-

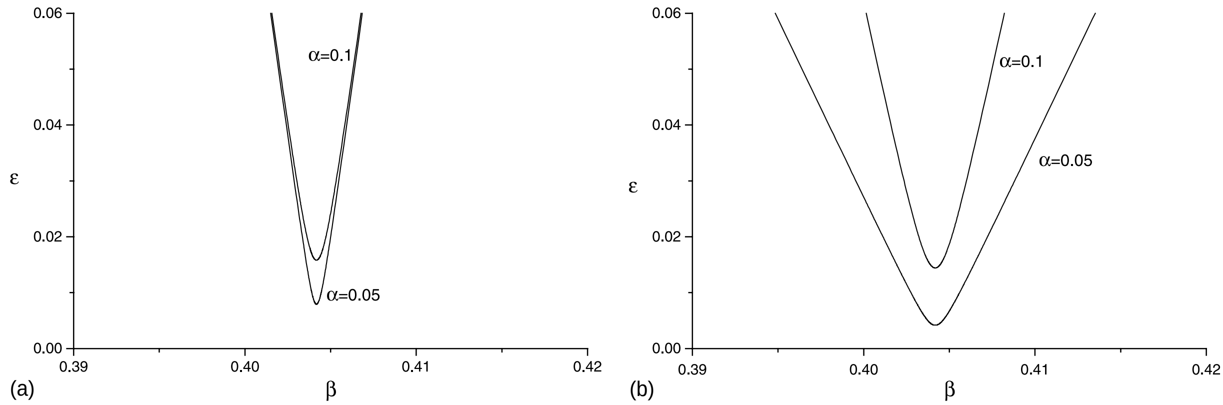

Fig. 2. The effect of the viscous damping on the stability boundaries of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the first main resonance $\beta \approx 2\omega_1$. $L/b = 10$, $h/b = 0.25$, $\gamma = 90°$, $\Omega_0 = 0.1$, $P_0 = 0.5$, $S = 0.2$. (a) First-moment stability boundaries; (b) second-moment stability boundaries.

bility boundary is more remarkable than on the first-moment stability boundary. Therefore, the viscous damping is favorable to the stability of the pre-twisted beam in this problem. Furthermore, the unstable region predicted by the first-moment stability criterion is smaller than and lies inside the unstable region predicted by the second-moment stability criterion; therefore, the first-moment stability criterion is less conservative than the second-moment stability criterion.

Fig. 3 depicts the effect of the total pre-twisted angle γ on the second-moment stability boundaries of the system. One finds that unstable regions exist only at main resonances but not at sum-type or difference-type resonances. As the angle γ increases, the locations of the first and third main resonances move towards the higher frequency domain, but that of the second main resonance moves towards the lower frequency domain. Moreover, the size of the unstable regions reduces gradually for higher main resonances. Note that the height of the tip of an unstable region is a good measure of the size of it. The lower the height is, the larger the unstable region is, and vice versa. Fig. 4 illustrates the heights of the tips of these unstable regions with respect to the total pre-twisted angle. As the angle γ increases, the height of the tip of the unstable region at the first main resonance goes upwards; the height of the tip of the unstable region at the second main resonance goes downwards, while that of the unstable region at the third main resonance goes upwards first and then falls down when the angle exceeds 130°. These phenomena relate excellently to the trends of changes of the first three natural frequencies of the pre-twisted beam against increases in the pre-twisted angle.

The effect of the average spin rate Ω_0 on the second-moment stability boundaries of the system is presented in Fig. 5. Again unstable regions exist only at main resonances, and the locations of the first and third main resonances move towards the higher frequency domain, but that of the second main resonance moves towards the lower frequency domain as the average spin rate increases. However, all unstable regions become larger with an increase in the average spin rate. Consequently, the effect of the average spin rate is destabilizing. The heights of the tips of the unstable regions of the pre-twisted beam with respect to the average spin rate are shown in Fig. 6. As the average spin rate increases, the heights of the tips of all unstable regions fall down drastically, and the tips of all unstable regions touch the horizontal axis at $\Omega_0 = 0.191$, at which the pre-twisted beam is unstable when spinning with a constant rate and subjected to the axial random force (Young and Gau, 2003). The stability boundaries of the system do not exist any more for a further increase in Ω_0.

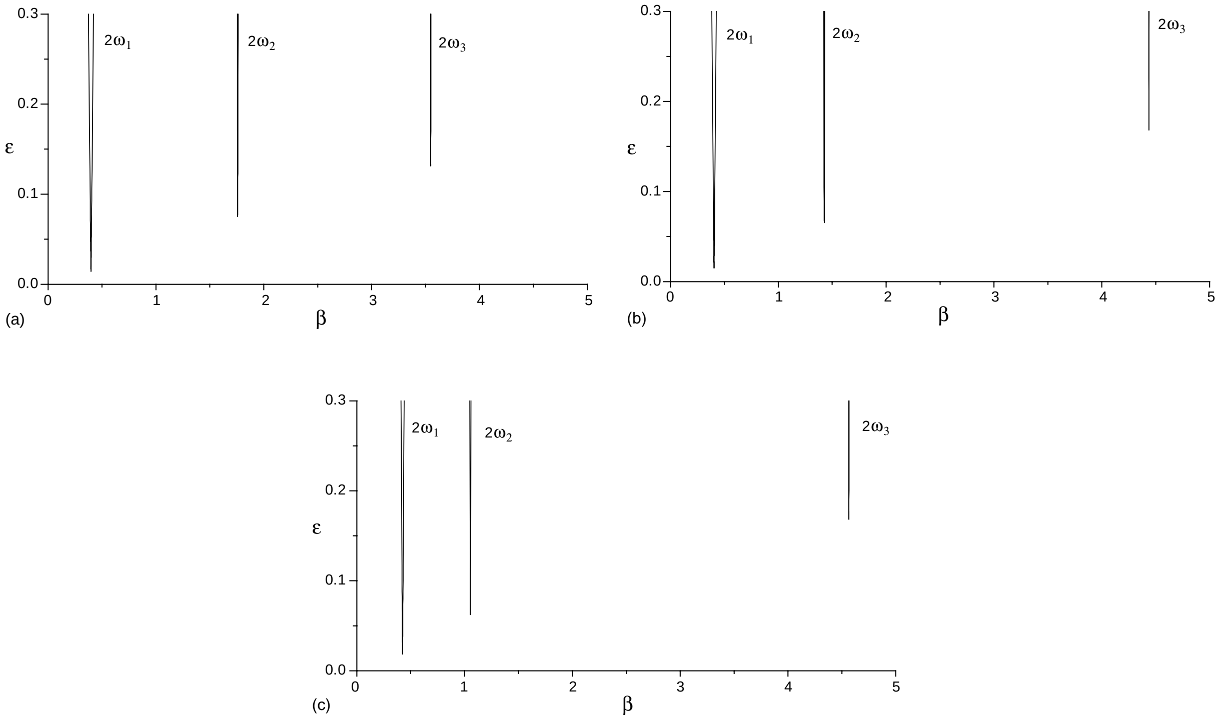

Fig. 3. The effect of the total pre-twisted angle on the second-moment stability boundaries of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end. $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\Omega_0 = 0.1$, $P_0 = 0.5$, $S = 0.2$. (a) $\gamma = 45°$; (b) $\gamma = 90°$; (c) $\gamma = 180°$.

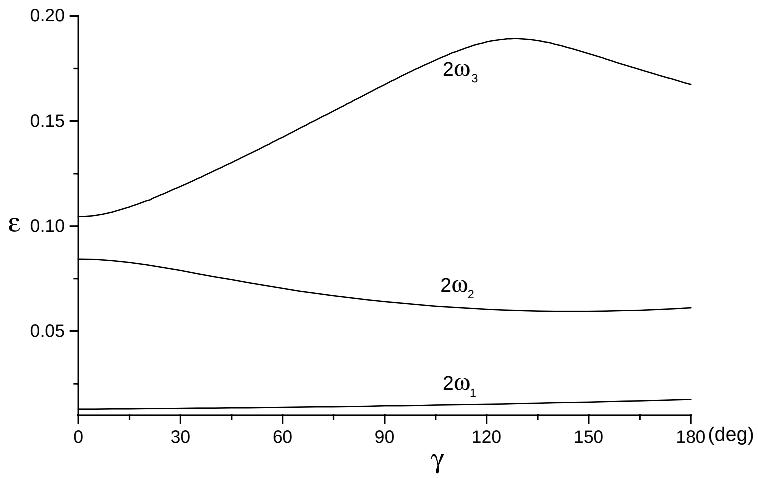

Fig. 4. The heights of the tips of the second-moment unstable regions of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end. $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\Omega_0 = 0.1$, $P_0 = 0.5$, $S = 0.2$.

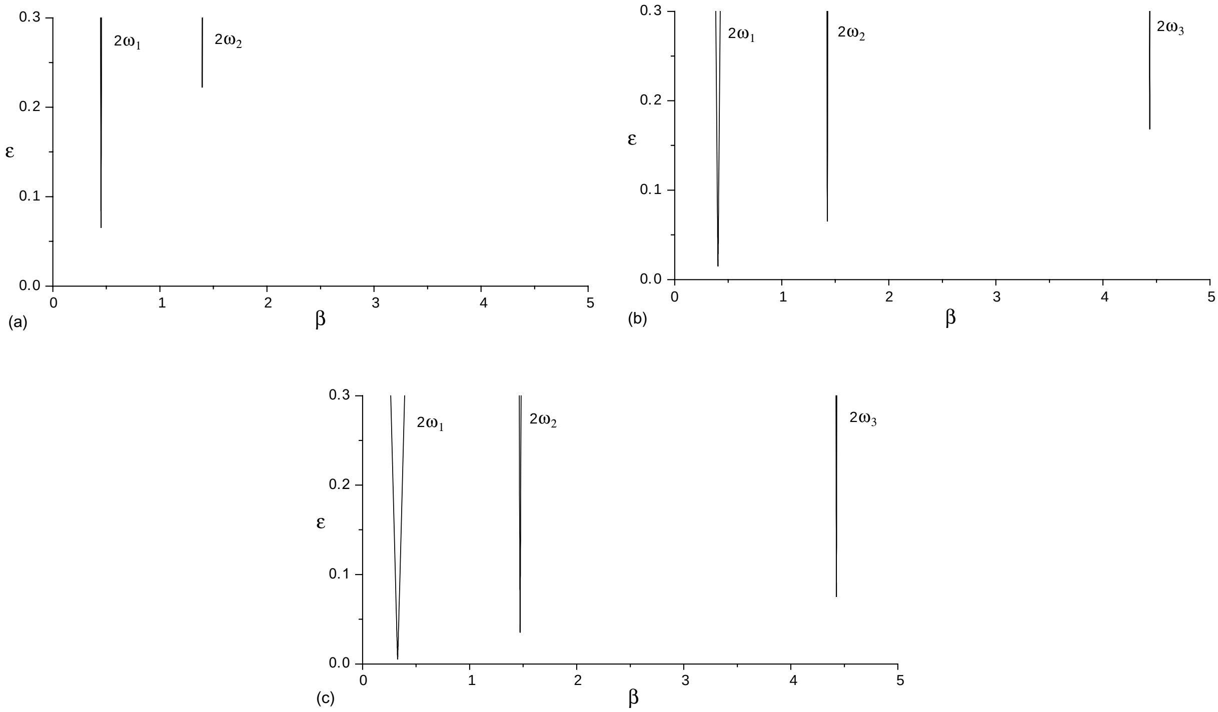

Fig. 5. The effect of the average spin rate on the second-moment stability boundaries of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end; $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\gamma = 90°$, $P_0 = 0.5$, $S = 0.2$. (a) $\Omega_0 = 0.05$; (b) $\Omega_0 = 0.1$; (c) $\Omega_0 = 0.15$.

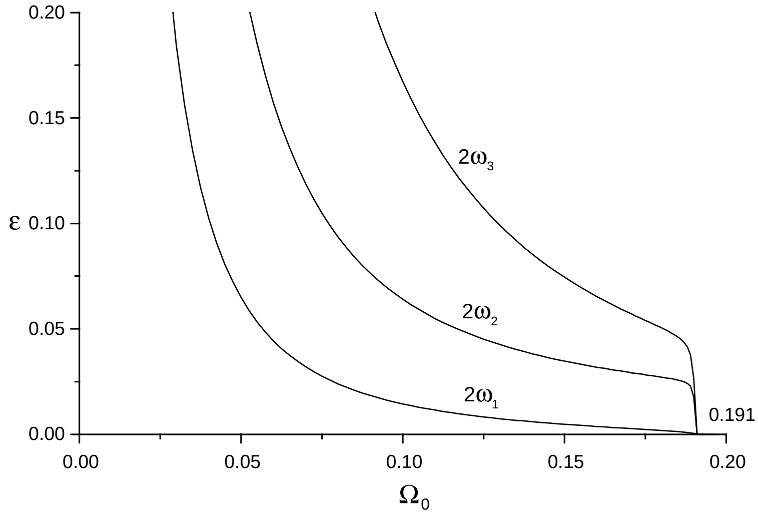

Fig. 6. The heights of the tips of the second-moment unstable regions of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end. $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\gamma = 90°$, $P_0 = 0.5$, $S = 0.2$.

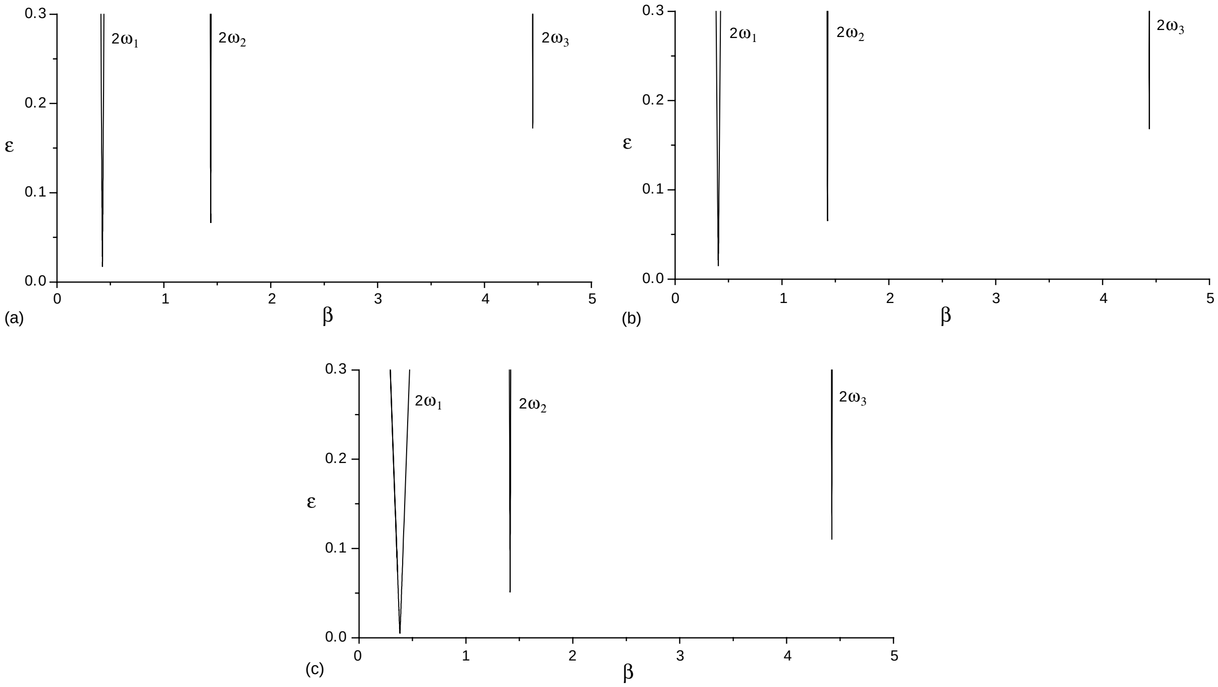

Fig. 7. The effect of the average axial force on the second-moment stability boundaries of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end. $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\gamma = 90°$, $\Omega_0 = 0.1$, $S = 0.2$. (a) $P_0 = 0.3$; (b) $P_0 = 0.5$; (c) $P_0 = 0.7$.

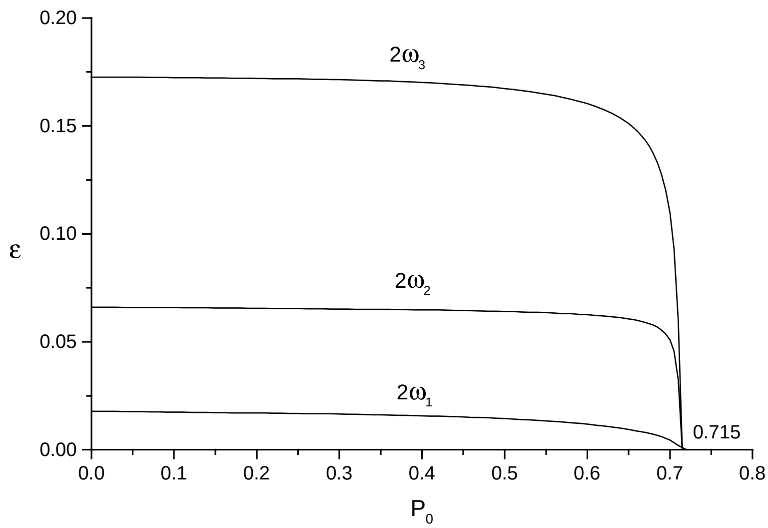

Fig. 8. The heights of the tips of the second-moment unstable regions of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end. $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\gamma = 90°$, $\Omega_0 = 0.1$, $S = 0.2$.

Fig. 7 depicts the effect of the average axial force P_0 on the second-moment stability boundaries of the system. All unstable regions get larger and move towards the lower frequency domain with an increase in the average axial force because an increase in the average axial force will reduce all natural frequencies of the pre-twisted beam. Consequently, the effect of the average axial force is destabilizing. Fig. 8 illustrates the heights of the tips of the unstable regions of the pre-twisted beam with respect to the average axial force. As the average axial force increases, the heights of the tips of all unstable regions fall down slowly and drop sharply as P_0 approaches 0.715, where the pre-twisted beam becomes unstable when spinning with a

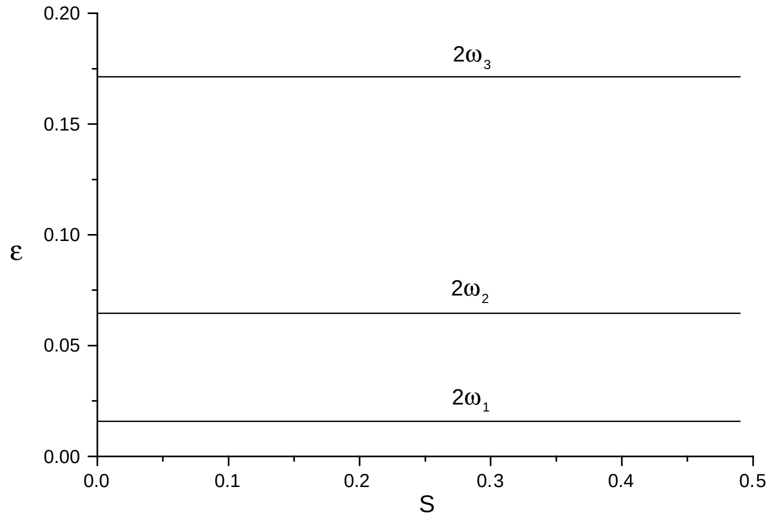

Fig. 9. The heights of the tips of the first-moment unstable regions of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end. $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\gamma = 90°$, $\Omega_0 = 0.1$, $P_0 = 0.5$.

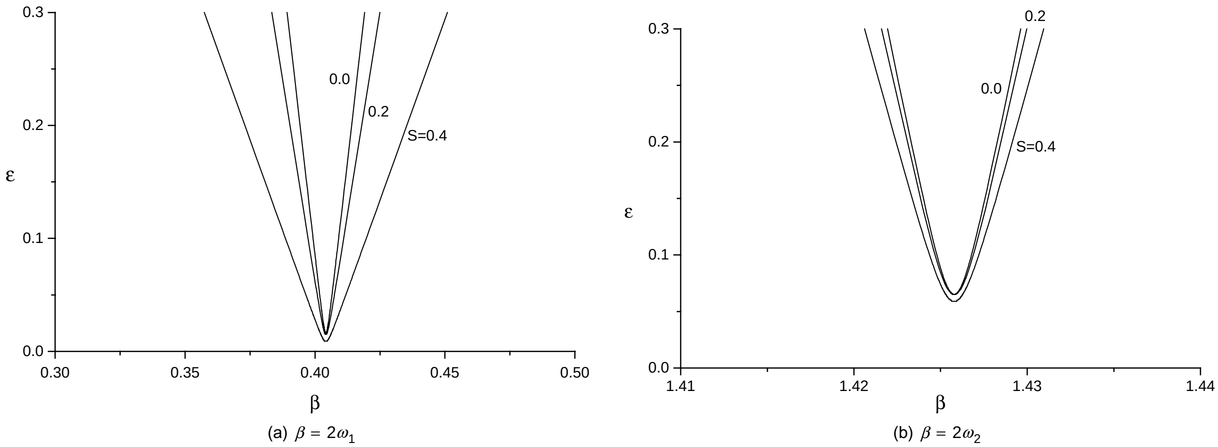

Fig. 10. The effect of the spectral density of the axial random force on the second-moment stability boundaries of a pre-twisted cantilever beam with a non-constant spin rate under an axial random force at the free end. $L/b = 10$, $h/b = 0.25$, $\alpha = 0.1$, $\gamma = 90°$, $\Omega_0 = 0.1$, $P_0 = 0.5$. (a) The first main resonance; (b) the second main resonance.

constant rate and subjected to the axial random force (Young and Gau, 2003). The stability boundaries of the system do not exist any more for a further increase in P_0.

The heights of the tips of the first-moment unstable regions of the pre-twisted beam with respect to the spectral density of the random part of the axial force are presented in Fig. 9. It is observed that the height of the tip of each unstable region is almost unchanged to the variation of the spectral density. Therefore, the effect of the spectral density on the first-moment stability of the system is insignificant. But the effect of the spectral density of the axial random force on the second-moment stability boundaries of the system is much more visible, as shown in Fig. 10. In this figure, all unstable regions lower and widen as the spectral density increases, and the effect on the first main resonance $\beta \approx 2\omega_1$ is more evident than on the second main resonance $\beta \approx 2\omega_2$. Therefore, the random part of the axial force is unfavorable to the second-moment stability of the system.

7. Conclusions

The dynamic stability of a pre-twisted cantilever beam spinning along its longitudinal axis with a non-constant spin rate and subjected to an axial random force at the free end is analyzed based on the theory of both deterministic and stochastic averaging. The spin rate of the beam is characterized as the sum of a constant (average) rate and a small, periodic perturbation function, and the axial force is assumed as the sum of a static (average) force and a weakly stationary random process with a zero mean. It is demonstrated that when the axial force is static, i.e., the beam is subjected to a deterministic parametric excitation only due to the periodically varying spin rate, both the first-moment and the second-moment stability criteria become identical and reduce to the result by Liao and Huang (1995b) at each resonant frequency combination. When the spin rate is constant, i.e., the beam is subjected to a parametric random excitation only due to the axial force, the first-moment stability criteria at different resonant frequency combinations reduce to the same form, so are the second-moment stability criteria, which are the same as the result obtained previously by the authors (2001).

As an application of the general solution, the spin rate perturbation is taken as a simple harmonic function, and the random part of the axial force is assumed as a Gaussian white noise process in the numerical illustrations. The effects of various system parameters on the first-moment and the second-moment stability boundaries of the beam are investigated, and the following conclusions can be drawn:

(1) When the beam is stable if it is acted upon by the axial random force only, the stability boundaries of the system when subjected to both deterministic (spin rate perturbation) and random (axial random force) excitations look like the stability boundaries acted by the deterministic parametric excitation only. When the beam is unstable if it is acted upon by the axial random force along, the beam is still unstable when subjected to both deterministic and random excitations.
(2) For a beam with a periodically varying spin rate under an axial random force, unstable regions exist only at main resonances but not at sum- or difference-type resonances, and the unstable regions reduce gradually for higher main resonances.
(3) The viscous damping is favorable to the stability of the beam, and its effect on the second-moment stability boundaries is more remarkable than on the first-moment stability boundaries. The random part of the axial force is unfavorable to the stability of the beam, and its effect on the first stability boundaries is insignificant.
(4) The effects of average spin rate and average axial force are destabilizing to the second-moment stability of the beam at all main resonances, but the effect of the total pre-twisted angle to the second-moment stability of the beam may be stabilizing or destabilizing at different main resonances.

Appendix A

$$[\sigma\sigma^{\mathrm{T}}]_{j,k} = \frac{\varepsilon}{4}\Big[(q_{jj}^{11}+q_{jj}^{22}+\mathrm{i}q_{jj}^{12}-\mathrm{i}q_{jj}^{21})(q_{kk}^{11}+q_{kk}^{22}+\mathrm{i}q_{kk}^{12}-\mathrm{i}q_{kk}^{21})z_j z_k S(0) + (q_{jk}^{11}+q_{jk}^{22}+\mathrm{i}q_{jk}^{12}-\mathrm{i}q_{jk}^{21})(q_{kj}^{11}+q_{kj}^{22}+\mathrm{i}q_{kj}^{12}-\mathrm{i}q_{kj}^{21}) \times z_j z_k S((\kappa_j-\kappa_k)\beta_s)\Big], \tag{A.1}$$

$$[\sigma\sigma^{\mathrm{T}}]_{j,k+N} = \frac{\varepsilon}{4}\Big[(q_{jj}^{11}+q_{jj}^{22}+\mathrm{i}q_{jj}^{12}-\mathrm{i}q_{jj}^{21})(q_{kk}^{11}+q_{kk}^{22}-\mathrm{i}q_{kk}^{12}+\mathrm{i}q_{kk}^{21})z_j \bar{z}_k S(0) + (q_{jk}^{11}-q_{jk}^{22}-\mathrm{i}q_{jk}^{12}-\mathrm{i}q_{jk}^{21})(q_{kj}^{11}-q_{kj}^{22}+\mathrm{i}q_{kj}^{12}+\mathrm{i}q_{kj}^{21}) \times z_j \bar{z}_k S((\kappa_j+\kappa_k)\beta_s)\Big] \quad k \neq j, \tag{A.2}$$

$$[\sigma\sigma^{\mathrm{T}}]_{j,j+N} = \frac{\varepsilon}{4}\left\{\sum_{r=1}^{N}\Big[(q_{jr}^{11}-q_{jr}^{22}-\mathrm{i}q_{jr}^{12}-\mathrm{i}q_{jr}^{21})(q_{jr}^{11}-q_{jr}^{22}+\mathrm{i}q_{jr}^{12}+\mathrm{i}q_{jr}^{21})\cdot z_r\bar{z}_r S((\kappa_r+\kappa_j)\beta_s) + (q_{jr}^{11}+q_{jr}^{22}+\mathrm{i}q_{jr}^{12}-\mathrm{i}q_{jr}^{21})(q_{jr}^{11}+q_{jr}^{22}-\mathrm{i}q_{jr}^{12}+\mathrm{i}q_{jr}^{21})\cdot z_r\bar{z}_r S((\kappa_r-\kappa_j)\beta_s)\Big]\right\}, \tag{A.3}$$

$$[\sigma\sigma^{\mathrm{T}}]_{j+N,k} = \overline{[\sigma\sigma^{\mathrm{T}}]}_{k,j+N}, \tag{A.4}$$

$$[\sigma\sigma^{\mathrm{T}}]_{j+N,k+N} = \frac{\varepsilon}{4}\Big[(q_{jj}^{11}+q_{jj}^{22}-\mathrm{i}q_{jj}^{12}+\mathrm{i}q_{jj}^{21})(q_{kk}^{11}+q_{kk}^{22}-\mathrm{i}q_{kk}^{12}+\mathrm{i}q_{kk}^{21})\bar{z}_j\bar{z}_k S(0) + (q_{jk}^{11}+q_{jk}^{22}-\mathrm{i}q_{jk}^{12}+\mathrm{i}q_{jk}^{21})(q_{kj}^{11}+q_{kj}^{22}-\mathrm{i}q_{kj}^{12}+\mathrm{i}q_{kj}^{21}) \times \bar{z}_j\bar{z}_k S((\kappa_j-\kappa_k)\beta_s)\Big], \tag{A.5}$$

$$\begin{aligned} U_n(z_1\bar{z}_1,\ldots,z_N\bar{z}_N) = \varepsilon\bar{z}_n\Big\{ &-2\tilde{\alpha}(c_{nn}^{11}+c_{nn}^{22})z_n + \frac{1}{4}\sum_{r=1}^{N}\{[(q_{nr}^{11}+q_{nr}^{22})(q_{rn}^{11}+q_{rn}^{22})-(q_{nr}^{12}-q_{nr}^{21})(q_{rn}^{12}-q_{rn}^{21})]z_n S((\kappa_n-\kappa_r)\beta_s) \\ &+ [(q_{nr}^{12}-q_{nr}^{21})(q_{rn}^{11}+q_{rn}^{22})+(q_{nr}^{11}+q_{nr}^{22})(q_{rn}^{12}-q_{rn}^{21})]z_n\psi((\kappa_n-\kappa_r)\beta_s) \\ &+ [(q_{nr}^{11}-q_{nr}^{22})(q_{rn}^{11}-q_{rn}^{22})+(q_{nr}^{12}+q_{nr}^{21})(q_{rn}^{12}+q_{rn}^{21})]z_n S((\kappa_n+\kappa_r)\beta_s) \\ &+ [(q_{nr}^{12}+q_{nr}^{21})(q_{rn}^{11}-q_{rn}^{22})-(q_{nr}^{11}-q_{nr}^{22})(q_{rn}^{12}+q_{rn}^{21})]z_n\psi((\kappa_n+\kappa_r)\beta_s)\}\Big\} \\ &+ \frac{\varepsilon}{4}\sum_{r=1}^{N}\Big[(q_{nr}^{11}-q_{nr}^{22}-\mathrm{i}q_{nr}^{12}-\mathrm{i}q_{nr}^{21})(q_{nr}^{11}-q_{nr}^{22}+\mathrm{i}q_{nr}^{12}+\mathrm{i}q_{nr}^{21})z_r\bar{z}_r S((\kappa_n+\kappa_r)\beta_s) \\ &+ (q_{nr}^{11}+q_{nr}^{22}+\mathrm{i}q_{nr}^{12}-\mathrm{i}q_{nr}^{21})(q_{nr}^{11}+q_{nr}^{22}-\mathrm{i}q_{nr}^{12}+\mathrm{i}q_{nr}^{21})z_r\bar{z}_r S((\kappa_r-\kappa_n)\beta_s)\Big] \quad n=1,2,\ldots,N;\ \ n\neq p, \end{aligned} \tag{A.6}$$

$$U_{p1}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_p^2,\bar{z}_p^2)$$

$$= \varepsilon\Bigg\{ -2\tilde{\alpha}(c_{pp}^{11}+c_{pp}^{22})z_p\bar{z}_p - f_s(g_{pp}^{11}-g_{pp}^{22}-\mathrm{i}g_{pp}^{12}-\mathrm{i}g_{pp}^{21})\bar{z}_p^2$$

$$-\frac{1}{2}f_s\beta_s(\mathrm{i}h_{pp}^{11}-\mathrm{i}h_{pp}^{22}+h_{pp}^{12}+h_{pp}^{21})\bar{z}_p^2 - f_s(k_{pp}^{11}-k_{pp}^{22}-\mathrm{i}k_{pp}^{12}-\mathrm{i}k_{pp}^{21})\bar{z}_p^2 - \bar{f}_s(g_{pp}^{11}-g_{pp}^{22}+\mathrm{i}g_{pp}^{12}+\mathrm{i}g_{pp}^{21})z_p^2$$

$$-\frac{1}{2}\bar{f}_s\beta_s(-\mathrm{i}h_{pp}^{11}+\mathrm{i}h_{pp}^{22}+h_{pp}^{12}+h_{pp}^{21})z_p^2 - \bar{f}_s(k_{pp}^{11}-k_{pp}^{22}+\mathrm{i}k_{pp}^{12}+\mathrm{i}k_{pp}^{21})z_p^2$$

$$+\frac{1}{4}\sum_{r=1}^{N}\Big\{[(q_{pr}^{11}+q_{pr}^{22})(q_{rp}^{11}+q_{rp}^{22})-(q_{pr}^{12}-q_{pr}^{21})(q_{rp}^{12}-q_{rp}^{21})]z_p\bar{z}_pS((\kappa_p-\kappa_r)\beta_s)$$

$$+[(q_{pr}^{12}-q_{pr}^{21})(q_{rp}^{11}+q_{rp}^{22})+(q_{pr}^{11}+q_{pr}^{22})(q_{rp}^{12}-q_{rp}^{21})]z_p\bar{z}_p\psi((\kappa_p-\kappa_r)\beta_s)$$

$$+[(q_{pr}^{11}-q_{pr}^{22})(q_{rp}^{11}-q_{rp}^{22})+(q_{pr}^{12}+q_{pr}^{21})(q_{rp}^{12}+q_{rp}^{21})]z_p\bar{z}_pS((\kappa_p+\kappa_r)\beta_s)$$

$$+[(q_{pr}^{12}+q_{pr}^{21})(q_{rp}^{11}-q_{rp}^{22})-(q_{pr}^{11}-q_{pr}^{22})(q_{rp}^{12}+q_{rp}^{21})]z_p\bar{z}_p\psi((\kappa_p+\kappa_r)\beta_s)\Big\}\Bigg\}$$

$$+\frac{\varepsilon}{4}\sum_{r=1}^{N}\Big[(q_{pr}^{11}-q_{pr}^{22}-\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{pr}^{11}-q_{pr}^{22}+\mathrm{i}q_{pr}^{12}+\mathrm{i}q_{pr}^{21})z_r\bar{z}_rS((\kappa_p+\kappa_r)\beta_s)$$

$$+(q_{pr}^{11}+q_{pr}^{22}+\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{pr}^{11}+q_{pr}^{22}-\mathrm{i}q_{pr}^{12}+\mathrm{i}q_{pr}^{21})z_r\bar{z}_rS((\kappa_r-\kappa_p)\beta_s)\Big], \tag{A.7}$$

$$U_{p2}(z_p\bar{z}_p,z_p^2)$$

$$=2\varepsilon z_p\Bigg\{\mathrm{i}\omega_p\lambda z_p-\tilde{\alpha}(c_{pp}^{11}+c_{pp}^{22}+\mathrm{i}c_{pp}^{12}-\mathrm{i}c_{pp}^{21})z_p-f_s(g_{pp}^{11}-g_{pp}^{22}-\mathrm{i}g_{pp}^{12}-\mathrm{i}g_{pp}^{21})\bar{z}_p$$

$$-\frac{1}{2}f_s\beta_s(\mathrm{i}h_{pp}^{11}-\mathrm{i}h_{pp}^{22}+h_{pp}^{12}+h_{pp}^{21})\bar{z}_p-f_s(k_{pp}^{11}-k_{pp}^{22}-\mathrm{i}k_{pp}^{12}-\mathrm{i}k_{pp}^{21})\bar{z}_p$$

$$+\frac{1}{8}\sum_{r=1}^{N}\Big\{(q_{pr}^{11}q_{rp}^{11}+q_{pr}^{11}q_{rp}^{22}+q_{pr}^{22}q_{rp}^{11}+q_{pr}^{22}q_{rp}^{22}-q_{pr}^{12}q_{rp}^{12}+q_{pr}^{12}q_{rp}^{21}+q_{pr}^{21}q_{rn}^{12}-q_{pr}^{21}q_{rp}^{21})z_p[S((\kappa_p-\kappa_r)\beta_s)$$

$$-\mathrm{i}\psi((\kappa_p-\kappa_r)\beta_s)]+\mathrm{i}(q_{pr}^{11}q_{rp}^{11}-q_{pr}^{11}q_{rp}^{22}+q_{pr}^{22}q_{rp}^{11}-q_{pr}^{22}q_{rp}^{22}+q_{pr}^{12}q_{rp}^{12}+q_{pr}^{12}q_{rp}^{21}+q_{pr}^{21}q_{rn}^{12}+q_{pr}^{21}q_{rp}^{21})z_p$$

$$\times[S((\kappa_p-\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p-\kappa_r)\beta_s)]+(q_{pr}^{11}q_{rp}^{11}-q_{pr}^{11}q_{rp}^{22}-q_{pr}^{22}q_{rp}^{11}+q_{pr}^{12}q_{rp}^{12}+q_{pr}^{12}q_{rp}^{21}$$

$$+q_{pr}^{21}q_{rp}^{12}+q_{pr}^{22}q_{rp}^{22}q_{pr}^{21}q_{rp}^{21})z_p[S((\kappa_p+\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p+\kappa_r)\beta_s)]+\mathrm{i}(q_{pr}^{11}q_{rp}^{11}+q_{pr}^{11}q_{rp}^{22}-q_{pr}^{22}q_{rp}^{11}$$

$$-q_{pr}^{22}q_{rp}^{22}-q_{pr}^{12}q_{rp}^{12}+q_{pr}^{12}q_{rp}^{21}-q_{pr}^{21}q_{rp}^{12}+q_{pr}^{21}q_{rp}^{21})z_p[S((\kappa_p+\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p+\kappa_r)\beta_s)]\Big\}\Bigg\}$$

$$+\frac{\varepsilon}{2}\Big[(q_{pp}^{11}+q_{pp}^{22}+\mathrm{i}q_{pp}^{12}-\mathrm{i}q_{pp}^{21})(q_{pp}^{11}+q_{pp}^{22}+\mathrm{i}q_{pp}^{12}-\mathrm{i}q_{pp}^{21})z_p^2S(0)\Big], \tag{A.8}$$

$$
\begin{aligned}
&U_{s1}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_pz_q,\bar{z}_p\bar{z}_q)\\
&\quad=\varepsilon\Bigg\{-2\tilde{\alpha}(c^{11}_{pp}+c^{22}_{pp})z_p\bar{z}_p-f_s(g^{11}_{pq}-g^{22}_{pq}-\mathrm{i}g^{12}_{pq}-\mathrm{i}g^{21}_{pq})\bar{z}_p\bar{z}_q\\
&\qquad-\frac{1}{2}f_s\beta_s(\mathrm{i}h^{11}_{pq}-\mathrm{i}h^{22}_{pq}+h^{12}_{pq}+h^{21}_{pq})\bar{z}_p\bar{z}_q-f_s(k^{11}_{pq}-k^{22}_{pq}-\mathrm{i}k^{12}_{pq}-\mathrm{i}k^{21}_{pq})\bar{z}_p\bar{z}_q\\
&\qquad-\bar{f}_s(g^{11}_{pq}-g^{22}_{pq}+\mathrm{i}g^{12}_{pq}+\mathrm{i}g^{21}_{pq})z_pz_q-\frac{1}{2}\bar{f}_s\beta_s(-\mathrm{i}h^{11}_{pq}+\mathrm{i}h^{22}_{pq}+h^{12}_{pq}+h^{21}_{pq})z_pz_q\\
&\qquad-f_s(k^{11}_{pq}-k^{22}_{pq}+\mathrm{i}k^{12}_{pq}+\mathrm{i}k^{21}_{pq})z_pz_q+\frac{1}{4}\sum_{r=1}^{N}\Big\{[(q^{11}_{pr}+q^{22}_{pr})\\
&\qquad\times(q^{11}_{rp}+q^{22}_{rp})-(q^{12}_{pr}-q^{21}_{pr})(q^{12}_{rp}-q^{21}_{rp})]z_p\bar{z}_pS((\kappa_p-\kappa_r)\beta_s)+[(q^{12}_{pr}-q^{21}_{pr})(q^{11}_{rp}+q^{22}_{rp})\\
&\qquad+(q^{11}_{pr}+q^{22}_{pr})(q^{12}_{rp}-q^{21}_{rp})]z_p\bar{z}_p\psi((\kappa_p-\kappa_r)\beta_s)+[(q^{11}_{pr}-q^{22}_{pr})(q^{11}_{rp}-q^{22}_{rp})+(q^{12}_{pr}+q^{21}_{pr})\\
&\qquad\times(q^{12}_{rp}+q^{21}_{rp})]z_p\bar{z}_pS((\kappa_p+\kappa_r)\beta_s)+[(q^{12}_{pr}+q^{21}_{pr})(q^{11}_{rp}-q^{22}_{rp})\\
&\qquad-(q^{11}_{pr}-q^{22}_{pr})(q^{12}_{rp}+q^{21}_{rp})]z_p\bar{z}_p\psi((\kappa_p+\kappa_r)\beta_s)\Big\}\Bigg\}\\
&\qquad+\frac{\varepsilon}{4}\sum_{r=1}^{N}\Big[(q^{11}_{pr}-q^{22}_{pr}-\mathrm{i}q^{12}_{pr}-\mathrm{i}q^{21}_{pr})(q^{11}_{pr}-q^{22}_{pr}+\mathrm{i}q^{12}_{pr}+\mathrm{i}q^{21}_{pr})z_r\bar{z}_rS((\kappa_p+\kappa_r)\beta_s)\\
&\qquad+(q^{11}_{pr}+q^{22}_{pr}+\mathrm{i}q^{12}_{pr}-\mathrm{i}q^{21}_{pr})(q^{11}_{pr}+q^{22}_{pr}-\mathrm{i}q^{12}_{pr}+\mathrm{i}q^{21}_{pr})z_r\bar{z}_rS((\kappa_r-\kappa_p)\beta_s)\Big],
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
&U_{s2}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_pz_q,\bar{z}_p\bar{z}_q)\\
&\quad=\varepsilon\Bigg\{-2\tilde{\alpha}(c^{11}_{qq}+c^{22}_{qq})z_q\bar{z}_q-\bar{f}_s(g^{11}_{qp}-g^{22}_{qp}-\mathrm{i}g^{12}_{qp}-\mathrm{i}g^{21}_{qp})\bar{z}_p\bar{z}_q\\
&\qquad-\frac{1}{2}\bar{f}_s\beta_s(\mathrm{i}h^{11}_{qp}-\mathrm{i}h^{22}_{qp}+h^{12}_{qp}+h^{21}_{qp})\bar{z}_p\bar{z}_q-\bar{f}_s(k^{11}_{qp}-k^{22}_{qp}-\mathrm{i}k^{12}_{qp}-\mathrm{i}k^{21}_{qp})\bar{z}_p\bar{z}_q\\
&\qquad-f_s(g^{11}_{qp}-g^{22}_{qp}+\mathrm{i}g^{12}_{qp}+\mathrm{i}g^{21}_{qp})z_pz_q-\frac{1}{2}f_s\beta_s(-\mathrm{i}h^{11}_{qp}+\mathrm{i}h^{22}_{qp}+h^{12}_{qp}+h^{21}_{qp})z_pz_q\\
&\qquad-f_s(k^{11}_{qp}-k^{22}_{qp}+\mathrm{i}k^{12}_{qp}+\mathrm{i}k^{21}_{qp})z_pz_q\\
&\qquad+\frac{1}{4}\sum_{r=1}^{N}\Big\{[(q^{11}_{qr}+q^{22}_{qr})(q^{11}_{rq}+q^{22}_{rq})-(q^{12}_{qr}-q^{21}_{qr})(q^{12}_{rq}-q^{21}_{rq})]z_q\bar{z}_qS((\kappa_q-\kappa_r)\beta_s)\\
&\qquad+[(q^{12}_{qr}-q^{21}_{qr})(q^{11}_{rq}+q^{22}_{rq})+(q^{11}_{qr}+q^{22}_{qr})(q^{12}_{rq}-q^{21}_{rq})]z_q\bar{z}_q\psi((\kappa_q-\kappa_r)\beta_s)\\
&\qquad+[(q^{11}_{qr}-q^{22}_{qr})(q^{11}_{rq}-q^{22}_{rq})+(q^{12}_{qr}+q^{21}_{qr})(q^{12}_{rq}+q^{21}_{rq})]z_q\bar{z}_qS((\kappa_q+\kappa_r)\beta_s)\\
&\qquad+[(q^{12}_{qr}+q^{21}_{qr})(q^{11}_{rq}-q^{22}_{rq})-(q^{11}_{qr}-q^{22}_{qr})(q^{12}_{rq}+q^{21}_{rq})]z_q\bar{z}_q\psi((\kappa_q+\kappa_r)\beta_s)\Big\}\Bigg\}\\
&\qquad+\frac{\varepsilon}{4}\sum_{r=1}^{N}\Big[(q^{11}_{qr}-q^{22}_{qr}-\mathrm{i}q^{12}_{qr}-\mathrm{i}q^{21}_{qr})(q^{11}_{qr}-q^{22}_{qr}+\mathrm{i}q^{12}_{qr}+\mathrm{i}q^{21}_{qr})z_r\bar{z}_rS((\kappa_q+\kappa_r)\beta_s)\\
&\qquad+(q^{11}_{qr}+q^{22}_{qr}+\mathrm{i}q^{12}_{qr}-\mathrm{i}q^{21}_{qr})(q^{11}_{qr}+q^{22}_{qr}-\mathrm{i}q^{12}_{qr}+\mathrm{i}q^{21}_{qr})z_r\bar{z}_rS((\kappa_r-\kappa_q)\beta_s)\Big],
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
&U_{s3}(z_p\bar{z}_p, z_q\bar{z}_q, z_pz_q)\\
&= \varepsilon\Big\{ \mathrm{i}(\omega_p+\omega_q)\lambda z_pz_q - \tilde{\alpha}(c_{pp}^{11}+c_{pp}^{22}-c_{qq}^{11}-c_{qq}^{22})z_pz_q - f_s(g_{pq}^{11}-g_{pq}^{22}-\mathrm{i}g_{pq}^{12}-\mathrm{i}g_{pq}^{21})z_q\bar{z}_q\\
&\quad - f_s(g_{qp}^{11}-g_{qp}^{22}-\mathrm{i}g_{qp}^{12}-\mathrm{i}g_{qp}^{21})z_p\bar{z}_p - \frac{1}{2}f_s\beta_s(\mathrm{i}h_{pq}^{11}-\mathrm{i}h_{pq}^{22}+h_{pq}^{12}+h_{pq}^{21})z_q\bar{z}_q\\
&\quad - \frac{1}{2}f_s\beta_s(\mathrm{i}h_{qp}^{11}-\mathrm{i}h_{qp}^{22}+h_{qp}^{12}+h_{qp}^{21})\cdot z_p\bar{z}_p - f_s(k_{pq}^{11}-k_{pq}^{22}-\mathrm{i}k_{pq}^{12}-\mathrm{i}k_{pq}^{21})z_q\bar{z}_q\\
&\quad - f_s(k_{qp}^{11}-k_{qp}^{22}-\mathrm{i}k_{qp}^{12}-\mathrm{i}k_{qp}^{21})z_p\bar{z}_p + \frac{1}{8}\sum_{r=1}^{N}[(q_{pr}^{11}+q_{pr}^{22}+\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{rp}^{11}+q_{rp}^{22}+\mathrm{i}q_{rp}^{12}-\mathrm{i}q_{rp}^{21})\\
&\quad \times z_pz_q(S((\kappa_p-\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p-\kappa_r)\beta_s)) + (q_{pr}^{11}-q_{pr}^{22}-\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})\\
&\quad \times (q_{rp}^{11}-q_{rp}^{22}+\mathrm{i}q_{rp}^{12}+\mathrm{i}q_{rp}^{21})z_pz_q(S((\kappa_p+\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p+\kappa_r)\beta_s))]\\
&\quad + \frac{1}{8}\sum_{r=1}^{N}[(q_{qr}^{11}+q_{qr}^{22}+\mathrm{i}q_{qr}^{12}-\mathrm{i}q_{qr}^{21})(q_{rq}^{11}+q_{rq}^{22}+\mathrm{i}q_{rq}^{12}-\mathrm{i}q_{rq}^{21})z_pz_q(S((\kappa_q-\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_q-\kappa_r)\beta_s))\\
&\quad + (q_{qr}^{11}-q_{qr}^{22}-\mathrm{i}q_{qr}^{12}-\mathrm{i}q_{qr}^{21})(q_{rq}^{11}-q_{rq}^{22}+\mathrm{i}q_{rq}^{12}+\mathrm{i}q_{rq}^{21})z_pz_q(S((\kappa_q+\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_q+\kappa_r)\beta_s))]\Big\}\\
&\quad + \frac{\varepsilon}{4}\Big[(q_{pp}^{11}+q_{pp}^{22}+\mathrm{i}q_{pp}^{12}-\mathrm{i}q_{pp}^{21})(q_{qq}^{11}+q_{qq}^{22}+\mathrm{i}q_{qq}^{12}-\mathrm{i}q_{qq}^{21})z_pz_qS(0)\\
&\quad + (q_{pq}^{11}+q_{pq}^{22}+\mathrm{i}q_{pq}^{12}-\mathrm{i}q_{pq}^{21})(q_{qp}^{11}+q_{qp}^{22}+\mathrm{i}q_{qp}^{12}-\mathrm{i}q_{qp}^{21})z_pz_qS((\kappa_p-\kappa_q)\beta_s)\Big],
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
&U_{d1}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N, z_q\bar{z}_q, \bar{z}_pz_q)\\
&= \varepsilon\Big\{ -2\tilde{\alpha}(c_{pp}^{11}+c_{pp}^{22})z_p\bar{z}_p - f_s(g_{pq}^{11}+g_{pq}^{22}+\mathrm{i}g_{pq}^{12}-\mathrm{i}g_{pq}^{21})\bar{z}_pz_q - \frac{1}{2}f_s\beta_s(\mathrm{i}h_{pq}^{11}+\mathrm{i}h_{pq}^{22}-h_{pq}^{12}+h_{pq}^{21})\bar{z}_pz_q\\
&\quad - f_s(k_{pq}^{11}+k_{pq}^{22}+\mathrm{i}k_{pq}^{12}-\mathrm{i}k_{pq}^{21})\bar{z}_pz_q - \bar{f}_s(g_{pq}^{11}+g_{pq}^{22}-\mathrm{i}g_{pq}^{12}+\mathrm{i}g_{pq}^{21})z_p\bar{z}_q\\
&\quad - \frac{1}{2}\bar{f}_s\beta_s(-\mathrm{i}h_{pq}^{11}-\mathrm{i}h_{pq}^{22}-h_{pq}^{12}+h_{pq}^{21})z_p\bar{z}_q - \bar{f}_s(k_{pq}^{11}+k_{pq}^{22}-\mathrm{i}k_{pq}^{12}+\mathrm{i}k_{pq}^{21})z_p\bar{z}_q\\
&\quad + \frac{1}{4}\sum_{r=1}^{N}\Big\{[(q_{pr}^{11}+q_{pr}^{22})(q_{rp}^{11}+q_{rp}^{22})-(q_{pr}^{12}-q_{pr}^{21})(q_{rp}^{12}-q_{rp}^{21})]z_p\bar{z}_pS((\kappa_p-\kappa_r)\beta_s)\\
&\quad + [(q_{pr}^{12}-q_{pr}^{21})(q_{rp}^{11}+q_{rp}^{22})+(q_{pr}^{11}+q_{pr}^{22})(q_{rp}^{12}-q_{rp}^{21})]z_p\bar{z}_p\psi((\kappa_p-\kappa_r)\beta_s)\\
&\quad + [(q_{pr}^{11}-q_{pr}^{22})(q_{rp}^{11}-q_{rp}^{22})+(q_{pr}^{12}+q_{pr}^{21})(q_{rp}^{12}+q_{rp}^{21})]z_p\bar{z}_pS((\kappa_p+\kappa_r)\beta_s)\\
&\quad + [(q_{pr}^{12}+q_{pr}^{21})(q_{rp}^{11}-q_{rp}^{22})-(q_{pr}^{11}-q_{pr}^{22})(q_{rp}^{12}+q_{rp}^{21})]z_p\bar{z}_p\psi((\kappa_p+\kappa_r)\beta_s)\Big\}\Big\}\\
&\quad + \frac{\varepsilon}{4}\sum_{r=1}^{N}\Big[(q_{pr}^{11}-q_{pr}^{22}-\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{pr}^{11}-q_{pr}^{22}+\mathrm{i}q_{pr}^{12}+\mathrm{i}q_{pr}^{21})z_r\bar{z}_rS((\kappa_p+\kappa_r)\beta_s)\\
&\quad + (q_{pr}^{11}+q_{pr}^{22}+\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{pr}^{11}+q_{pr}^{22}-\mathrm{i}q_{pr}^{12}+\mathrm{i}q_{pr}^{21})z_r\bar{z}_rS((\kappa_r-\kappa_p)\beta_s)\Big],
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
&U_{d2}(z_1\bar{z}_1,\ldots,z_N\bar{z}_N,z_p\bar{z}_q,\bar{z}_pz_q)\\
&= \varepsilon\Bigg\{-2\tilde{\alpha}(c_{qq}^{11}+c_{qq}^{22})z_q\bar{z}_q-\bar{f}_s(g_{qp}^{11}+g_{qp}^{22}+\mathrm{i}g_{qp}^{12}-\mathrm{i}g_{qp}^{21})z_p\bar{z}_q+\frac{1}{2}\bar{f}_s\beta_s(\mathrm{i}h_{qp}^{11}+\mathrm{i}h_{qp}^{22}-h_{qp}^{12}+h_{qp}^{21})z_p\bar{z}_q\\
&\quad-\bar{f}_s(k_{qp}^{11}+k_{qp}^{22}+\mathrm{i}k_{qp}^{12}-\mathrm{i}k_{qp}^{21})z_p\bar{z}_q-f_s(g_{qp}^{11}+g_{qp}^{22}-\mathrm{i}g_{qp}^{12}+\mathrm{i}g_{qp}^{21})\bar{z}_pz_q+\frac{1}{2}f_s\beta_s(-\mathrm{i}h_{qp}^{11}-\mathrm{i}h_{qp}^{22}-h_{qp}^{12}\\
&\quad+h_{qp}^{21})\bar{z}_pz_q-f_s(k_{qp}^{11}+k_{qp}^{22}-\mathrm{i}k_{qp}^{12}+\mathrm{i}k_{qp}^{21})\bar{z}_pz_q+\frac{1}{4}\sum_{r=1}^{N}\Big\{[(q_{qr}^{11}+q_{qr}^{22})(q_{rq}^{11}+q_{rq}^{22})-(q_{qr}^{12}-q_{qr}^{21})(q_{rq}^{12}\\
&\quad-q_{rq}^{21})]z_q\bar{z}_qS((\kappa_q-\kappa_r)\beta_s)+[(q_{qr}^{12}-q_{qr}^{21})(q_{rq}^{11}+q_{rq}^{22})+(q_{qr}^{11}+q_{qr}^{22})(q_{rq}^{12}-q_{rq}^{21})]z_q\bar{z}_q\psi((\kappa_q-\kappa_r)\beta_s)\\
&\quad+[(q_{qr}^{11}-q_{qr}^{22})(q_{rq}^{11}-q_{rq}^{22})+(q_{qr}^{12}+q_{qr}^{21})(q_{rq}^{12}+q_{rq}^{21})]z_q\bar{z}_qS((\kappa_q+\kappa_r)\beta_s)+[(q_{qr}^{12}+q_{qr}^{21})(q_{rq}^{11}-q_{rq}^{22})\\
&\quad-(q_{qr}^{11}-q_{qr}^{22})(q_{rq}^{12}+q_{rq}^{21})]z_q\bar{z}_q\psi((\kappa_q+\kappa_r)\beta_s)\Big\}\Bigg\}+\frac{\varepsilon}{4}\sum_{r=1}^{N}\Big[(q_{qr}^{11}-q_{qr}^{22}-\mathrm{i}q_{qr}^{12}-\mathrm{i}q_{qr}^{21})(q_{qr}^{11}-q_{qr}^{22}+\mathrm{i}q_{qr}^{12}\\
&\quad+\mathrm{i}q_{qr}^{21})z_r\bar{z}_rS((\kappa_q+\kappa_r)\beta_s)+(q_{qr}^{11}+q_{qr}^{22}+\mathrm{i}q_{qr}^{12}-\mathrm{i}q_{qr}^{21})(q_{qr}^{11}+q_{qr}^{22}-\mathrm{i}q_{qr}^{12}+\mathrm{i}q_{qr}^{21})z_r\bar{z}_rS((\kappa_r-\kappa_q)\beta_s)\Big],
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
&U_{d3}(z_p\bar{z}_p,z_q\bar{z}_q,z_p\bar{z}_q)\\
&= \varepsilon\Bigg\{+\mathrm{i}(\omega_p-\omega_q)\lambda z_p\bar{z}_q-\tilde{\alpha}(c_{pp}^{11}+c_{pp}^{22}+c_{qq}^{11}+c_{qq}^{22})z_p\bar{z}_q-f_s(g_{pq}^{11}+g_{pq}^{22}+\mathrm{i}g_{pq}^{12}-\mathrm{i}g_{pq}^{21})z_q\bar{z}_q\\
&\quad-f_s(g_{qp}^{11}+g_{qp}^{22}-\mathrm{i}g_{qp}^{12}+\mathrm{i}g_{qp}^{21})z_p\bar{z}_p-\frac{1}{2}f_s\beta_s(\mathrm{i}h_{pq}^{11}+\mathrm{i}h_{pq}^{22}-h_{pq}^{12}+h_{pq}^{21})z_q\bar{z}_q\\
&\quad-\frac{1}{2}f_s\beta_s(-\mathrm{i}h_{qp}^{11}-\mathrm{i}h_{qp}^{22}-h_{qp}^{12}+h_{qp}^{21})\cdot z_p\bar{z}_p-f_s(k_{pq}^{11}+k_{pq}^{22}+\mathrm{i}k_{pq}^{12}-\mathrm{i}k_{pq}^{21})z_q\bar{z}_q\\
&\quad-f_s(k_{qp}^{11}+k_{qp}^{22}-\mathrm{i}k_{qp}^{12}+\mathrm{i}k_{qp}^{21})z_p\bar{z}_p+\frac{1}{8}\sum_{r=1}^{N}[(q_{pr}^{11}+q_{pr}^{22}+\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})(q_{rp}^{11}+q_{rp}^{22}+\mathrm{i}q_{rp}^{12}-\mathrm{i}q_{rp}^{21})z_p\bar{z}_q\\
&\quad\times(S((\kappa_p-\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p-\kappa_r)\beta_s))+(q_{pr}^{11}-q_{pr}^{22}-\mathrm{i}q_{pr}^{12}-\mathrm{i}q_{pr}^{21})\\
&\quad\times(q_{rp}^{11}-q_{rp}^{22}+\mathrm{i}q_{rp}^{12}+\mathrm{i}q_{rp}^{21})z_p\bar{z}_q(S((\kappa_p+\kappa_r)\beta_s)-\mathrm{i}\psi((\kappa_p+\kappa_r)\beta_s))]\\
&\quad+\frac{1}{8}\sum_{r=1}^{N}\Big[(q_{qr}^{11}+q_{qr}^{22}-\mathrm{i}q_{qr}^{12}+\mathrm{i}q_{qr}^{21})(q_{rq}^{11}+q_{rq}^{22}-\mathrm{i}q_{rq}^{12}+\mathrm{i}q_{rq}^{21})z_p\bar{z}_q(S((\kappa_q-\kappa_r)\beta_s)+\mathrm{i}\psi((\kappa_q-\kappa_r)\beta_s))\\
&\quad+(q_{qr}^{11}-q_{qr}^{22}+\mathrm{i}q_{qr}^{12}+\mathrm{i}q_{qr}^{21})(q_{rq}^{11}-q_{rq}^{22}-\mathrm{i}q_{rq}^{12}-\mathrm{i}q_{rq}^{21})z_p\bar{z}_q(S((\kappa_q+\kappa_r)\beta_s)\\
&\quad+\mathrm{i}\psi((\kappa_q+\kappa_r)\beta_s))\Big]\Bigg\}+\frac{\varepsilon}{4}\Big[(q_{pp}^{11}+q_{pp}^{22}+\mathrm{i}q_{pp}^{12}-\mathrm{i}q_{pp}^{21})(q_{qq}^{11}+q_{qq}^{22}-\mathrm{i}q_{qq}^{12}+\mathrm{i}q_{qq}^{21})z_p\bar{z}_qS(0)\\
&\quad+(q_{pq}^{11}-q_{pq}^{22}-\mathrm{i}q_{pq}^{12}-\mathrm{i}q_{pq}^{21})(q_{qp}^{11}-q_{qp}^{22}+\mathrm{i}q_{qp}^{12}+\mathrm{i}q_{qp}^{21})z_p\bar{z}_qS((\kappa_p+\kappa_q)\beta_s)\Big].
\end{aligned}
\tag{A.14}
$$

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